Consider the uncertainty about whether it will rain in Brisbane next weekend. A weather forecaster may be able to assess a precise probability of rain, such as 0.3285..., although even an expert should feel uncomfortable about specifying a probability to more than one or two decimal places. Someone who has little information about the prospects for rain may be able to make only an imprecise judgment such as “it will probably not rain”, or “it is more likely to rain tomorrow than at the weekend”, or “the probability of rain is between 0.2 and 0.4”. People living outside Australia may be completely ignorant about the weather in Brisbane and assign lower probability 0 and upper probability 1. Probabilities based on extensive data can be distinguished, through their precision, from those based on ignorance.

As a simple statistical example, consider an urn containing coloured balls. Initially nothing is known about the colours. How should we model the uncertainty about the colour of the next ball that will be drawn from the urn, and how should we update the model after some balls are sampled? Intuitively, because we are completely ignorant about the colours initially, all conceivable colours should be assigned lower probability 0 and upper probability 1. After repeated sampling with replacement, the posterior upper and lower probabilities of any colour should become increasingly precise and converge to the observed relative frequency of that colour. There are imprecise probability models for the learning process which have these properties, which treat all colours symmetrically, and which are coherent.

Imprecise probability models are needed in many applications of probabilistic and statistical reasoning. They have been used in the following kinds of problems:

- when there is little information on which to evaluate a probability, as in Walley [35, 37, 38]
- to model nonspecific information, e.g., knowing the proportions of black, white and coloured balls in an urn gives only upper and lower bounds for the chance of drawing a red ball (see Dempster [6], Klir and Folger [23] and Shafer [30])
- to model the uncertainty produced by vague statements such as “it will probably rain” or “there is a good chance that it will be mainly fine” (Walley [36], Zadeh [46])
- in robust Bayesian inference, to model uncertainty about a prior distribution (see Berger [1] for a survey, and also Berger and Berliner [2], DeRobertis and Hartigan [8], Pericchi and Walley [29])
- to model conflict between several sources of information, e.g., disagreement between the probability judgments of experts or between prior information and statistical data (for practical applications see Walley et al. [39, 42])
in frequentist studies of robustness, to allow imprecision in a statistical sampling model, e.g., data from a normal distribution may be contaminated by a few outliers or errors that come from a completely unknown distribution (Huber [21])

to model physical processes which appear to be stationary but which produce unstable relative frequencies or unstable time averages (Fine [13], Walley and Fine [41])

to make probabilistic predictions about future observations, using either frequentist criteria (Hampel [19]) or principles of coherence (Walley [37])

to account for the ways in which people make decisions when they have indeterminate or ambiguous information (for surveys see Einhorn and Hogarth [10] and Smithson [34]).

Mathematical models

Imprecise probability is used as a generic term to cover all mathematical models which measure chance or uncertainty without sharp numerical probabilities. It includes both qualitative and imprecise or nonadditive quantitative models. Most probability judgments in everyday life are qualitative, involving terms like “probably” and “more likely than”, rather than numerical. There is a large literature on the following types of imprecise probability model:

1. comparative probability orderings (“it is more likely to rain tomorrow than the next day”) (Keynes [22], Koopman [24], Fine [11, 12], Fishburn [14])
2. upper and lower probabilities, also called “interval-valued” or “nonadditive” probabilities (Smith [33], Kyburg [26], Good [17]). Models 3–6 are special types of lower or upper probability:
3. belief functions (Dempster [6], Shafer [30, 31])
4. Choquet capacities (Choquet [4], Huber [20, 21], Denneberg [7])
5. fuzzy measures (Klir and Folger [23], de Campos et al. [3], Wang and Klir [43])
6. possibility measures (Zadeh [45], Dubois and Prade [9], Klir and Folger [23])
7. sets of probability measures (Levi [28], Berger [1])
8. upper and lower previsions (Williams [44], Walley [35, 36]).

There are many other models which allow imprecision, including classificatory models (“probably it will not rain”) (Fine [11], Walley and Fine [40]), sets of desirable gambles (Williams [44], Walley [35]), partial preference orderings (Giles [15], Giron and Rios [16], Walley [35]), intervals of measures (DeRobertis and Hartigan [8]) and Baconian probabilities (Cohen [5]).

Relationships between the models

There is evidently a wide variety of mathematical models for imprecise probability. However, the differences between these models are not as great as they may appear to be. From a mathematical point of view, all the models listed above are equivalent to special kinds of upper or lower previsions. Upper and lower previsions can therefore provide a general framework for a theory of imprecise probabilities.

Most research on imprecise probabilities has been concerned with types of upper and lower probability. However, it is important to recognise that some common kinds of uncertainty cannot be modelled adequately by upper and lower probabilities. For example, the comparative probability judgment “event $A$ is at least as probable as $B$” can be modelled in terms of lower previsions by $P(I_A - I_B) \geq 0$, but this constraint cannot be expressed in terms of upper and lower probabilities. Here $I_A$ denotes the indicator function of $A$, which takes the value 1 if $A$ occurs and takes the value 0 if $A$ does not occur.
In standard probability theory, no information is lost by specifying uncertainty in terms of a probability measure, because an additive probability measure determines unique previsions (expectations) and unique conditional probabilities (provided the conditioning event has non-zero probability). But information may be lost when uncertainty is modelled in terms of upper and lower probabilities, because they do not determine upper and lower previsions and conditional probabilities uniquely. Belief functions, Choquet capacities, fuzzy measures and possibility measures, which are types of upper or lower probability, are therefore not sufficiently general, although they are useful models in many applications.

Upper and lower previsions do seem to be sufficiently general to model all the common types of uncertainty. Upper and lower probabilities are special types of upper and lower previsions, defined only for indicator functions of events. Choquet capacities and fuzzy measures are general types of upper or lower probability which are required only to assign values 0 and 1 to the empty set and full sample space and to be monotone with respect to set inclusion. Choquet capacities of order 2 satisfy an additional property of 2-monotonicity, which guarantees that they are coherent lower probabilities. Choquet capacities of order infinity satisfy a stronger property of complete monotonicity and are more commonly known as belief functions. Thus belief functions are a special type of coherent lower probability. They can also be characterised as the lower probability functions that can be generated from a probability measure through a multivalued mapping. The corresponding upper probabilities are called plausibility functions. Possibility measures are plausibility functions which satisfy an additional property of consonance, which requires the upper probability of any set to be the supremum of the upper probabilities of its elements. Thus possibility measures are a special kind of coherent upper probability.

The other models listed above can also be identified with upper or lower previsions. Sets of probability measures generate upper and lower previsions as their upper and lower envelopes. An interval of measures is a special type of set of probability measures. The classificatory judgment “event $A$ is probable” is equivalent to the lower probability judgment $P(A) \geq 0.5$. Baconian probabilities are equivalent to possibility measures. Sets of desirable gambles and partial preference orderings are essentially equivalent, and they generate upper and lower previsions in a natural way. However, partial preference orderings and sets of desirable gambles are slightly more general and more informative than upper and lower previsions, e.g., they can be used to define upper and lower probabilities conditional on an event of upper probability zero, and these models deserve much more attention than they have received.

Despite the mathematical relationships between the models, there are differences in interpretation amongst the various theories of imprecise probability and each theory has its own distinctive flavour. Most studies have been based on a subjective interpretation of probabilities as degrees of belief, but there are also substantial bodies of work on imprecise logical probabilities (see Keynes [22], Kyburg [26, 27], Levi [28], Walley [37]), and imprecise physical probabilities (Huber [21], Walley and Fine [41], Fine [13] and Walley [35]). I believe that all the mathematical models listed above can be useful in particular kinds of applications. Readers should look elsewhere in this web site and consult the following references to learn more about the models.

**References**

For overviews of the topic, see especially Smithson [34], Walley [35] and Smets *et al.* [32]. For an introduction to the theory of lower previsions and comparison with belief functions and possibility measures, see Walley [36]. There have been...
many applications of imprecise probability models in other disciplines, including expert systems and artificial intelligence (see Krause and Clark [25] and Walley [36]), philosophy of science (Good [18], Kyburg [27], Levi [28]), experimental psychology (how does imprecision influence choices?) (Einhorn and Hogarth [10]), legal reasoning (how should a judge weigh legal arguments?) (Cohen [5]), the economics of energy systems (should a consumer invest in a solar heating system?) (Walley and Campello de Souza [39]), and medical data analysis (if the early evidence from a randomised clinical trial favours one treatment, is it ethical to continue randomising patients to other treatments?) (Walley et al. [37, 42]).