Formalisms For Fusion Belief in Design

Michele Pappalardo

DIMEC-Department of Mechanical Engineering
University of Salerno, Italy

Abstract

The nature and scope of fusion of imprecise data in the design is explored and discussed with subjective view of de Finetti. Every stage in the design process includes a level of approximation and is typically high in the preliminary phase. Engineering projects often are modified in several design iterations before being completed. Information received from many groups or experts working on a project will often necessitate changes in a design. The interaction between different groups associated with a design project often takes the form of an informal fusion of data. This form of interaction commonly arises when engineering information is imprecise. For the fusion of belief the theory of Dempster and Shafer is used. DS theory generalises the Bayesian rule and provides the mechanism for the fusion of information.

1. INTRODUCTION

The analysis of belief is a method for manipulating information on preliminarily designs. It is possible to combine preference using the aggregation of two or more design function. In design the subjective interpretation of Carnap, probability as measure degree of belief, is very often supported. The probability of a statement is the degree of confirmation on one event $A$. If the posterior probability $P^{new}(A)$ is greater or less of the prior $P^{old}(A)$ probability than we have the measure of degree of confirmation of Carnap

\[ \text{Measure of Carnap} = P^{new}(A) - P^{old}(A) \]

It is a measure for confirming or infirming a hypothesis. In Bayesian approach, let $A$ be a proposition the set of possible truths is $\{A, \neg A\}$, with the axiom of probability we have

\[ P(A) + P(\neg A) = 1 \]

Denoting various propositions $A, B, C$, etc., if the propositions $A \cap B$ is true and $\neg A$ is false than we have the Laplace’s mathematical representation of process of learning (Bayes Theorem):

\[ P(A|B, C) = P(B|A, C)P(A|C)/P(B|C) \]

The Bayes Theorem is an algorithm for update results and for acquiring new information. $P(A|C)$ is the prior probability for $A$ when we know only $C$ and $P(A|B, C)$ is the posterior information and updated as a results acquiring new information $B$. $A$ represents the hypothesis under analysis $C$ represents what we know about $A$ (table of truth) before getting $B$
(new data). The original degree of belief is replaced by a new degree of belief when new evidence is obtained.

Bayes’ theorem is simple a rule for manipulating probabilities but it cannot by itself help us to assign the probabilities. When the details of a design are unknown and there is not sufficient information, the preliminary decisions are the most important. The Laplace’s “Principle of Insufficient Reason” asserts that, when one has not sufficient information to distinguish between the beliefs of a lot of events, the best strategy is to consider all cases equally likely. The Laplace’s principle, mathematical translation of an ancient principle of wisdom, is a decision-making principle since assign values of belief.

One basic problem in making decision is how to evaluate the belief of an uncertain event. Since different hypothesis assign different distributions of belief, the problem is how to take in account different hypothesis and to assign only one final set of proper values which can came from probability, possibility or theory of evidence or from more variety of interpretations of probability have been proposed. In every design we can in situations where the information is incomplete and the truth-value of proposition is indeterminate. We must decide what is possible true and possible false and to do next on basis of a plausible reasoning without to lack information. According to N. Wiener in the learning the information results fundamental on the action in progress since it results so possible the action of feed-back. In the choice of the following action the principle of the feedback means that the behavior is compared periodically with the result from to achieve, and that the success or the failure of this behavior it modifies the future result. The comparison is founded on the measure of the information that is founded in turn on the value of the probabilities. For having the control of uncertainty, it is necessary to measure information on it. We can use as weight of evidence the Turing’s definition of information. If $x$ is a finite partition of a probability space, than the information function of $x$ is a step function whose value on an element of $x$ is the negative of the logarithm of the probability of this element (entropy). If we assume that the entropy of a system, in a given state, is directly proportional to the logarithm of the probability of finding it in that state, than entropy is the measure of the information of the system. The Shannon’s definition of entropy is

$$H(p(x_1),..., p(x_n)) = -\sum_{i=1}^{n} p(x_{i}) \log p(x_{i})$$

under the constraint $\sum_{i=1}^{n} p(x_{i}) = 1$. When the probability is equally likely the value of entropy is max. The entropy is the measure of uncertainty. When we have new results with new information, the new effect can be evaluated measuring the distance $D(p;q)$ between the distribution a priori distribution $q$ from the new distribution $p$ obtained using new data. The measure of distance can be calculated using the Kulback’s formula of cross-entropy.
\[ D(p : q) = \sum p \ln(p / q). \] If our \textit{a priori} distribution \( B(A) = 1 - B(\neg A) = 0.5 \) is of max entropy than values of cross-entropy are equal to Shannon’s values of entropy. Valid solution has minimum entropy. A system can have infinite solution consistent with given constraint, but only with maximum entropy, or minimum cross-entropy, probabilistic entities adjust their values as to give the optimum solution (MaxEnt principle of E.T.Jaynes). The Maximum Entropy principle assigns the probabilities. For to have an ideal solution, with minimum entropy, a system must depend from a finite and limited number of parameter. In a design we can have fair solutions if it is observed the fundamental axiom of Soft Design: \textit{Valid design has minimum values of information and depends on a finite and limited number of independent, or soft dependent, parameters.} The restrictions of the axiom involve reproducibility and stability of the same system. For to select a final valid proper selection, among possible solutions, it is necessary to fix a \textit{weighting parameter}. The \textit{real art is choosing an appropriate space of possibilities.} MaxEnt assign a prior probability distribution over this space. For minimizing expected loss, a Designer starts selecting a subset from the prior plausible data so that efficient processing is possible, and for to get a posterior plausible subset. \textit{The critical moment is the final selection from solutions consistent with given constraint.}

2. **INFECTION**

Probabilistic models use the probability distributions for to describe uncertainties. It is a good practice to construct probabilistic model of uncertainties using subjective information because designers claim that \textit{probability does not only express the frequency of an event but it also describes a subjective strength of belief that the event can occur.} One can use probabilistic models when has sufficient numerical data to estimate distribution of random variable. In absence of sufficient data the probabilistic model for design can be constructed on belief of experts. Bayesians believe that it is possible to make probability assignment even in presence of subjective degree of belief and in absence of frequency information.

The probability is central to the modeling of engineering phenomena and at the same time there is much disagreement about interpretation that should be attached to concept. We use two best-known interpretations: the \textit{classical} of Bernoulli and Laplace and the \textit{subjective} of de Finetti. In the classical interpretation an uncertain event can be decompose into equally likely cases and the probability is the ratio of favorable to total cases. The principle of Laplace asserts that, when, on an uncertain event one has not sufficient information to distinguish between the beliefs of the cases, the best strategy is to consider all equally likely.

With subjective view of de Finetti, the probability is the subjective strength of belief that can be assigned to any event repeatable or not. In subjective belief if \( p \) is the probability of an event \( E \) if \( S \) is betting amount (positive or negative) \textit{than} the gain will be \( (1-p)S \). The condition of \textit{coherence of de Finetti} is that gain \( (1-p)S \) and bet \( pS \) will be always of opposite sign. The mathematical translation is

\[ -p(1-p)S^2 \leq 0 \]

\( S^2 \) is always positive than will be \( p(1-p) \leq 0 \). This condition of coherence is always valid when is \( 0 \leq p \leq 1 \).

In the analysis of a design, if \( G \) is the gain and \( C \) is the loss (or cost), on the basis of MaxEnt principle, the relations between gain and loss give method for to find values of belief useful for to get valid solutions. Indicating with \( (C/G) = \gamma > 0 \), the expected utility relation of \textit{von Neumann-Morgenstern}, between gain and cost on event \( A \), is

\[ B(A)G - (1 - B(A))C = B(A) - (1 - B(A))\gamma = k \]
Analyzing the coherence of expected utility relation using the subjective definition of probability of de Finetti we must have:

\[ T = [-B(A)(1-B(A))] \leq 0 \]

The coherence is possible only if it is

\[ 0 \leq B(A) \leq 1 \]

In total absence of information the Laplacian strategy is to consider equally likely \( G, C, B(A) \) and \( B(\neg A) \). Then if \( G=C \) than is \( \gamma = 1 \) and MaxEnt principle gives the Laplacian distribution \( B(A) = 1 - B(\neg A) = 0.5 \) and \( k = 0 \). The Laplacian solution of is coherent with principle of de Finetti

\[ T = [-B(A)(1-B(A))] \leq -0.25 \leq 0 \]

With constant expected utility \( k \) for each modification of \( \gamma \), we get a new distribution. \( B(A) \) represents learning and tells us how update belief when our state of knowledge change trough acquisition of new values of \( \gamma \).

\[ B(A) = (\gamma + k) / (1 + \gamma) \]

The new effect can be evaluated measuring the measure of Carnap as distance \( D(p||q) \) between the Laplacian distribution \( q \) (a priori distribution) from the new distribution \( p \) obtained using new data.

The condition of coherence on the expected utility relation of von Neumann-Morgenstern can be obtained applying the condition of Laplace. The principle of Laplace asserts that when in a utility relation, has not sufficient information to distinguish between the beliefs of the cases, the best strategy is to consider all equally likely the gain and the bet, than

\[ B(A) = (1 - B(A)) \gamma ; \quad k = 0 ; \quad B(A) = \gamma / (1 + \gamma) \]

The value of \( \gamma \) is always \( \geq 0 \) and the range of belief is \( 0 \leq B(A) \leq 1 \). The condition of coherence gives the range of belief. The values of belief based on a priori distribution with \( k=0 \) is the best procedure for to obtain inference. With expected utility \( k=0 \), for each modification of \( \gamma \), we get a new distribution and \( B(A) \) is the Break Value of belief. We define the minimum measure of belief for having expected utility non negative as the break values \( BV \) is. The value of \( k = 0 \) in the expected utility relation of von Neumann-Morgenstern is the same condition of coherence of de Finetti on the gain and bet. The break value tells us how update belief when our state of knowledge change trough acquisition of new values of \( \gamma \).
It is possible to find from $\gamma$ a convex distributions of minimum measures of the belief. The degree of truth, in his conventional meaning of degree of plausibility and truth (but also of satisfaction) can be evaluated using the DS rules.

The intersection of information’s sets (assuming that the information is, for definition, true) tends to increase the degree of truth for which an hypothesis is definitely or confirmed or denied. The lack of convergence of the analysis means that it is not verified the premise of the process. The DS theory gives a method for to have the selection, among design alternatives, with greater belief. From the sets of break values the inference is

$$\{ Bel_i(A) \oplus Bel_j(A) \oplus Bel_k(A) \} \oplus ...$$

For the mass $m_i$ is

$$m_i(\{A\}) + m_i(\{\neg A\}) + m_i(\{A,\neg A\}) = 1$$

3. BELIEF

In inference problems the belief can be represented by mathematical entities $Bel(.) \in [0,1]$ with the sum to its negation equal to 1. In Bayesian approach on an probabilistic event $A$ the belief is $Bel(A)$ and $Bel(\neg A)$ if uncertainty is equal to zero than

$$Bel(\neg A) = 1 - Bel(A)$$

Using the belief we wants to allow to more general approach to representing uncertainty than Bayesian approach, thus if $Bel(A)$ is the mass of $A$ does not mean that $Bel(\neg A)$ in the weight of belief in $\neg A$. In our framework the measures correspond to the belief function. Using belief we have more general measure then probability and every event $A$ can lie in an interval. We can take belief and plausibility as lower and upper bounds of the probability assigned to an event. A method for handling data in presence of uncertainty with qualitative values is theory of Dempster-Shafer. The DS theory of is a method for reasoning under uncertainty, include Bayesian probability as special case, and introduce the belief function as lower probabilities...
and the plausibility function as upper probabilities. Here we are interested in applying theory because the numerical information required by Bayesian methods are not available. Numerical measure in presence of uncertainty may be assigned to set of propositions as well as single proposition. The probabilities are apportioned to subsets and the mass $\nu$ can move over each element. Let the finite non empty set $\Theta = \{x_1, \ldots, x_n\}$ be the frame of discernment which is the set of all hypothesis. The basic probability is assigned in the range $[0, 1]$ to the $2^n$ subset of $\Theta$ consisting of a singleton or conjunction of singleton of $n$ elements $x_i$. The basic probability is a function which assign the weigh to the subset such that $
olimits\sum_{A_\in \Theta} m(A) = 1 \quad m(\Phi) = 0$

The lower probability $P_\ast(A_j)$ is defined as $P_\ast(A_j) = \sum_{A_i \subset A_j} m(A_j)$

And the upper probability $P^\ast(A_j)$is defined as $P^\ast(A_j) = 1 - \sum_{A_i \subset A_j} m(A_j)$

The $m(A_j)$ values are the independent basic values of probability inferred on each subset $A_j$. The belief function of set $M$ if is given by $Bel(M) = \sum_{A_i \subset M} m(A_i)$ $Pl(M) = \sum_{A_i \cap M = \emptyset} m(A_i)$.

The evidential interval that provides a lower and upper bound is

$$Evidential - Interval = [Bl(M), Pl(M)]$$

If $m_1$ and $m_2$ are the independent basic probabilities from independent evidence, and $\{A_{i1}\}$ and $\{A_{i2}\}$ the sets of focal points, then the theorem of Shafer gives the rule of combination.

Let $m_1$ and $m_2$ two independent basic probabilities from independent evidence. If $\sum_{A_i \cap A_j = \emptyset} m_1(A_{i1}) m_2(A_{i2}) > 0$ then

$$m(A) = (m_1 \oplus m_2)(A) = \left\{ \begin{array}{l} \sum_{A_i \cap A_j = \emptyset} m_1(A_{i1}) m_2(A_{i2}) \quad A = \emptyset, m_1 \oplus m_2 = 2^n \rightarrow [0, 1] \\ 1 - \sum_{A_i \cap A_j = \emptyset} m_1(A_{i1}) m_2(A_{i2}) \quad A \neq \emptyset, (m_1 \oplus m_2) = 2^n \rightarrow [0, 1] \end{array} \right.$$

give the rule for combining two or more probability given from independent evidence.
If we assume that belief functions $B_l$, $B_{l_2}$, $B_{l_3}$, ..., $B_{l_n}$ are assigned in the same frame of discernment, than according with the Shafer’s rule of combination, the new belief function may be yielded via $Bel(A) = \{ [(Bel_1(A) \otimes Bel_2(A)) \otimes Bel_3(A)] \otimes ... \}$

### 4. APPLICATION

The example is a preliminary design task consisting in a selection between two solutions proposed from experts. The result of experts consist in two set of break values of belief. If the belief on two design is

<table>
<thead>
<tr>
<th>Solution 1</th>
<th>Expert E1</th>
<th>Expert E2</th>
<th>$m^{E1}(-A)$</th>
<th>$m^{E2}(-A)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solution 1</td>
<td>0.8</td>
<td>0.7</td>
<td>0.2</td>
<td>0.3</td>
</tr>
</tbody>
</table>

The solutions of experts is subjective and gives the mass on $m(\neg A)$ and $m(A, \neg A)$. For “Solution 1” the mass of belief are

$$\text{Solution 1} \quad m_{1.1}([A]) = 0.0, \quad m_{1.2}([\neg A]) = 0.06 + 0.14 + 0.24 = 0.44, \quad m_{1.3}([A, \neg A]) = 0.56$$

$$P_{\text{Solution 1}}([A]) = 1 \cdot m_{1.2}([\neg A]) = 0.56$$

For “Solution 2” the mass of belief are

$$\text{Solution 2} \quad m_{2.1}([A]) = 0.0, \quad m_{2.2}([\neg A]) = 0.1, \quad m_{2.3}([A, \neg A]) = 0.04 + 0.06 + 0.36 = 0.46$$

$$P_{\text{Solution 2}}([A]) = 1 \cdot m_{2.3}([A, \neg A]) = 0.54$$

We have

$$P_{\text{Solution 1}}([A]) > P_{\text{Solution 2}}([A])$$

“Solution 1” is more plausible than the “Solution 2”.
5. CONCLUSIONS

The nature and scope of fusion of imprecise data in engineering is explored and discussed with subjective view of de Finetti, and application of the method is illustrated. The intersection of information sets tends to increase the degree of truth for which an information is definitely either confirmed or denied. Preferences are expressed on an absolute scale, where the mass of beliefs indicates a completely acceptable value or completely unacceptable value. The fusion of design decision combines many individual preferences into a single, overall preference. The lack of convergence of the analysis means that has not verified the premise of the design.

Prof. MICHELE PAPPALARDO
DIMEC-Department of Mechanical Engineering
University of Salerno
84084 Fisciano (SA) - Italy
tel. +39 089 964326 - fax +39 089 964037
e-mail: pappalar@bridge.diima.unisa.it

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