Value at risk calculation through ARCH factor methodology: Proposal and comparative analysis

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Abstract

In this paper we put forward a new method to estimate value at risk (VaR), autoregressive conditional heteroskedastic (ARCH) factor, which combines multivariate analysis with ARCH models. Firstly, from a set of correlated portfolio risk factors, we derive a smaller uncorrelated risk factors set, by applying multivariate analysis. Secondly, we use ARCH schemes to model uncorrelated factors historical behaviour. Thirdly, we use the estimated models to predict future values for factors standard deviation. From them, VaR calculation is immediate. In this way, ARCH factor methodology overcomes the multivariate ARCH models drawbacks, which, in practice, make these unworkable for VaR calculation purposes. We apply the proposed methodology over a set of foreign exchange risk exposed portfolios, obtaining better results than those reached when J.P. Morgan’s Riskmetrics is used.

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1. Introduction

Value at risk (VaR) has become a key subject when quantifying portfolio market risk. VaR determines the maximum loss a portfolio can generate over a certain holding period; and this allows us to control the portfolio risk level at any time. Therefore, VaR can be used, for instance, in the evaluation of portfolio managers performance: it provides a risk quantification, expressed in monetary units (m.u.), which can be used jointly with portfolio return (also expressed in m.u.), to this end. Furthermore, it helps portfolio managers to determine the most adequate risk management policy for any situation. For this reason, VaR becomes specially interesting for financial institution supervisors and managers (see Basle Committee on Banking Supervision, 1996; Berger et al., 1995; Cordell and King, 1995; Dimson and Marsh, 1995, 1997; Gjerde and Semmen, 1995): from VaR quantification they can determine the amount of resources that these institutions need to immobilise as a guarantee against their risk exposure level.

Nevertheless, there is no single method for determining portfolio VaR. Thus, institutions and decision makers have a wide range of calculation methods when they want to estimate the VaR for a portfolio. It is convenient for them to evaluate different alternatives and select the most adequate
for their particular characteristics. That is why research on improvement of current methods and development of new ones becomes especially pertinent.

In this paper we put forward a new model to calculate the portfolio VaR: the autoregressive conditional heteroskedastic (ARCH) factor methodology. We implement this methodology using a set of market risk exposed portfolios, and we compare the results obtained with those provided by another methodology.

The rest of the paper is set out as follows: Section 2 introduces the main methodologies and models that can be used to determine VaR; Section 3 describes the ARCH factor methodology characteristics and particularities; in Section 4 we estimate the VaR on a set of foreign exchange risk exposed portfolios; Section 5 analyses the results obtained and compares them with those provided by the widely used J.P. Morgan Riskmetrics methodology; and the final section summarises the main conclusions of our study.

2. Value at risk calculation

VaR can be defined as “an estimate of the largest loss that a portfolio is likely to suffer during all but truly exceptional periods. More precisely, the VaR is the maximum loss that an institution can be confident it would lose (in) a certain fraction of time over a particular period” (Hopper, 1996). Thus, VaR sums up the risk which a portfolio or, in general, an entity, is exposed to in just one figure: the amount which could be lost during an established holding period. Moreover, VaR associates this amount with a determined statistic likelihood level (for an extensive review of VaR, see Duffie and Pan, 1997. Fong and Vasiceck, 1997; Stambaugh, 1996; Beder, 1995, can also be consulted).

Several methods can be used for portfolio VaR calculation. These methods are usually grouped into the following categories (Mori et al., 1996):

- historical simulation methods,
- Monte Carlo simulation methods,
- variance–covariance methods.

Within these methods, two alternative approaches can be used to quantify the portfolio VaR: the delta approach and the full valuation approach.

Historical and Monte Carlo simulation methods use the latter, while variance–covariance methods use the first approach (see Jorion, 1996; Longerstaey, 1996).

In historical simulation methods, an empirical distribution for the price changes over a period of time in the past is derived. VaR estimation arises from the maximum of the losses distribution, associated with the desired statistic likelihood percentile.

In Monte Carlo simulation methods, an empirical distribution for the price changes is derived. But this distribution must be generated from a set of pseudo-random variables, distributed in accordance with the assumed hypothesis (the most usual assumption is normality). From this distribution, VaR is determined following a parallel path to that used in historical simulation methods (Bauer, 2000; Khindanova and Rachev, 2000 provide VaR calculation methods without assuming normality).

Finally, in variance–covariance methods VaR is proportional to the portfolio standard deviation. Eq. (1) is usually used for VaR estimation:

$$\text{VaR}_t = \phi \sqrt{\sigma_{\text{ip}}^2}$$

where $\phi$ is the likelihood parameter; $\sigma_{\text{ip}}$ denotes the portfolio return standard deviation for time $t$; and $\tau$ is a parameter used when we calculate VaR for a time period with a different length from that used to estimate the portfolio standard deviation. For instance, when we want to calculate a 10 day VaR by using a daily standard deviation, $\tau$ equals 10.

The standard deviation of the portfolio return is quantified from past price change variances and covariances, for the relevant assets (the ones whose price changes affect the portfolio value). These changes are usually called risk factors. Eq. (2) summarises the calculation procedure required to estimate the portfolio return variance from its risk factor variances and covariances:

$$\sigma_{\text{ip}}^2 = \sum_{i=1}^{n} \delta_i^2 \sigma_{ii}^2 + 2 \sum \rho_{ij} \delta_i \delta_j \sigma_{ij} \sigma_{jl}$$


where $\delta_i$ denotes the portfolio return sensitivity to the $i$th risk factor; $\sigma_i t$ is the $i$th risk factor volatility for time $t$, quantified through its standard deviation; and $\rho_{ij}$ is the correlation coefficient between $i$th and $j$th risk factors. As we can clearly see in Eq. (2), the product between $i$th and $j$th risk factors volatilities and their correlation coefficient provides their covariance.

Within variance–covariance methods, several methodologies can be used to calculate the portfolio variance (about variance/volatility forecasting, Baillie and Bollerslev (1992), Jorion (1995), Brailsford and Faff (1996), Figlewski (1997), Amin and Ng (1997) can be consulted). A classification of these can be made as follows (see Hendricks, 1996; Hopper, 1996; Vasilellis and Meade, 1996; Mori et al., 1996):

- **Constant variance-covariance methodology**: This assumes that risk factor variances and covariances stay constant through time.

- **Equally weighted moving average methodology**: This leads away from the preceding hypothesis, but it assumes that forecasts about future variance–covariance values can be made by calculating them from a fixed amount of past data. This methodology equally weights all the past observations used, regardless of the time they correspond to.

- **Exponentially weighted moving average methodology**: This is the methodology used by Morgan and Reuters (1996) in their Riskmetrics proposal (see Phelan, 1997). The key element of the Riskmetrics method is a huge covariance matrix for the returns of a vast variety of assets, denominated in different currencies. The information from this matrix is freely provided by Morgan. As Riskmetrics is a widely used methodology, it is a natural point of reference for alternative models in the field.

The main difference between the exponentially and the equally weighted moving average approaches stems from the different weight associated to the past observations. Exponentially weighted moving average methodology emphasises recent observations by using exponentially weighted moving averages of squared deviations (3):

$$\sigma_i t = \sqrt{(1 - \lambda) \sum_{s=t-k}^{t-1} \lambda^{t-s-1}(x_{it} - \mu_{it})^2}$$

where $\sigma_i t$ denotes the $i$th predicted risk factor standard deviation for time $t$; $x_{it}$ denotes the $i$th risk factor value (price change) for time $s$; $\mu_{it}$ is the past average value for this risk factor; $k$ gives the number of observations included in the calculation; and parameter $\lambda$, referred to as “decay factor”, determines the rate at which the weights on past observations decay as they become older (in this paper we only show the equations for variance estimation; expressions for covariance estimations are similar).

The sum of observation weights in (3) are determined by (4):

$$\sum_{s=t-k}^{t-1} \lambda^{t-s-1} = (1 - \lambda)[1 + \lambda + \lambda^2 + \cdots + \lambda^{k-1}]$$

$$= \frac{(1 - \lambda)(1 - \lambda^k)}{(1 - \lambda)} = 1 - \lambda^k$$

considering that $0 < \lambda < 1$, when $k \to \infty$, $\lambda^k \to 0$. Thus, if we assume a large enough value for $k$, the sum of weights equals 1.

It is easily demonstrable that $i$th risk factor predicted standard deviation for time $t$ may be alternatively calculated from Eq. (5):

$$\sigma_i t \approx \sqrt{\lambda \sigma^2_{i(t-1)} + (1 - \lambda)(x_{i(t-1)} - \mu_{i(t-1)})^2},$$

i.e., from a linear combination between one period lagged variance value, and one period lagged squared deviation from risk factor average value. What Eq. (5) shows is a particular member of the ARCH models family.

- **ARCH models**: These models allow us to forecast future variance values by combining past squared deviations and past variance values. The original ARCH models were introduced by Engle (1982), and were generalised by Bollerslev (1986) a few years later (see Bollerslev et al. (1992) for further analysis of ARCH models). Bollerslev's model takes its starting point from a general regression model (6):
where $y_t$ denotes the dependent variable, and $X_t$ is the exogenous variables vector, which may include lagged values of the dependent variable; $\beta$ is the coefficients vector; and $\epsilon_t$ represents the stochastic error. Bollerslev's model characterises the $\epsilon_t$ distribution conditioned on the realised values of the set of exogenous variables ($\Psi_{t-1}$) as follows:

$$
ev_t/\Psi_{t-1} \sim N(0, h_t)$$

(7)

where the set of exogenous variables and realised values for the dependent variable ($\Psi_{t-1}$) is defined by

$$
\Psi_{t-1} = \{x_1, x_2, \ldots, x_m, x_{m-1}, \ldots, x_{m-p}, y_{t-1}, \ldots, y_{t-q}\}
$$

(8)

and normal distribution variance ($h_t$) can be expressed through Eq. (9):

$$
h_t = \alpha_0 + \alpha_1\epsilon_{t-1}^2 + \cdots + \alpha_q\epsilon_{t-q}^2 + \beta_1h_{t-1} + \cdots + \beta_p h_{t-p}
$$

(9)

This model (9) is known as a "GARCH (p,q) model", where $p$ denotes the number of considered lagged variance values and $q$ determines this number for the squared deviations. Within the VaR calculation field it can be used to make forecasts over future variance values. However, besides risk factor variance forecasts, covariance predictions are needed. In the ARCH models field, these variances and covariances must be simultaneously estimated using a multivariate ARCH model.

Within the VaR calculation field, we can represent a general multivariate ARCH model through expression (10):

$$
Y_t/\Psi_{t-1} \sim N(Z_t^\prime \Gamma, H_t)
$$

(10)

where $Y_t$ is a $N \times 1$ vector which collects the temporary series corresponding to the considered set of risk factors:

$$
Y_t = (y_1, \ldots, y_N).
$$

(11)

$Z_t^\prime \Gamma$ is a $N \times 1$ vector which collects the forecasts made for all of the aforementioned temporary series; and $H_t$ is the $N \times N$ matrix for the risk factors variances and covariances.

As we assumed, $H_t$ can be modelled through an ARCH scheme. Thus, we can represent this matrix as a function of the past squared forecasting errors and the conditional variances and covariances past values. If we use the vech notation (which stacks the lower triangular elements of a symmetric matrix in a column), we can represent $H_t$ in the following way:

$$
vech(H_t) = vech(\Sigma) + \sum_{i=1}^q A_i vech(\epsilon_{t-i}\epsilon_{t-i}^\prime) + \sum_{i=1}^p B_i vech(H_{t-i})
$$

(12)

where $\epsilon_t^\prime = (\epsilon_{t1}, \ldots, \epsilon_{tN})$ is the forecasting errors vector; $\Sigma$ is a positive definite $N \times N$ matrix, which collects the constant items of the model; and $A_i$ and $B_i$ are two $[N(N+1)/2 \times N(N+1)/2]$ matrices which collect the coefficients for, respectively, the squared forecasting errors and the lagged conditional variances and covariances. As we can see, expression (12) is a generalisation of expression (9) for multivariate models.

The problems of multivariate ARCH models lie in their estimation. Therefore, as Bera and Higgins (1993) state, some hypotheses are required when estimating a multivariate ARCH model, in order to guarantee a positive definite $H_t$ matrix. However their main drawback is the excessive number of parameters to calculate, due to the necessary joint estimation of conditional variances and covariances. In accordance with the general expression (12), the required number of parameters is defined by

$$
[N(N+1)/2][1 + [N(N+1)/2]][p + q].
$$

(13)

Therefore, as an example, if we consider only two risk factors, an estimation of $[12(p + q)]$ coefficients is needed within a multivariate ARCH scheme. For this reason, if no restrictive hypothesis is done, ARCH models cannot easily work for VaR calculation. Therefore papers that propose possible ways for introducing ARCH models in VaR calculation procedure present a specially interesting line of research (see Bollerslev and Engle, 1993). Barone-Adesi
et al. (1999), e.g., propose an historical simulation method that allows the introduction of ARCH models in VaR calculation procedure.

3. ARCH factor methodology definition

As stated above, the main drawback when using ARCH models for VaR calculation purposes stems from the compulsory joint estimation of risk factor variances and covariances. This gives rise to an unworkably high number of parameters to be estimated. The ARCH factor methodology, introduced in this paper, overcomes this drawback by using factor analysis techniques as a first step in the VaR calculation procedure (see Singh, 1997). These techniques allow us to generate a group of uncorrelated variables from a wider group of correlated variables (related to this, Zhu and Jamshidian (1997) propose a risk factor reduction technique but they do not relate it to ARCH models). Considering the uncorrelation property between the new variables, covariance estimations are not required. Therefore, risk factor variance estimations can be done individually.

ARCH factor methodology implementation requires a six stage procedure (preliminary versions of this scheme can be consulted at Cabedo and Moya, 1998, 1999):

1) Risk factors correlation analysis: The original set of variables (risk factors) are analysed, in order to determine whether they can be summarised into a reduced number of factors (multivariate factors; we use this denomination to differentiate them from risk factors). A series of statistical tests is carried out in order to establish the degree of correlation between risk factors. Usually Bartlett’s sphericity test and Kaiser–Meyer–Oklin’s (KMO) sampling adequacy test are used for this purpose.

Bartlett’s statistic tests the null hypothesis that the correlation coefficients for every couple of variables (risk factors) are equal to zero. This statistic is defined by

\[ x^2_{p,5(\kappa^2-K)} = \left[ n - 1 - \frac{1}{6}(2K + 5) \right] \ln|\mathbf{R}| \]  

(14)

where \( n \) is the number of observations; \( K \) is the considered number of variables (risk factors); and \(|\mathbf{R}|\) is the determinant for the residuals sample correlation matrix.

On the other hand, the KMO’s statistic is based upon the partial correlation coefficient between two variables (risk factors). This statistic must be calculated by applying expression (15):

\[ \text{KMO} = \frac{\sum_{i<j}||_{ij}^2} {\sum_{i,j} r_{ij}^2 + \sum_{i \neq h} \sum_{j \neq h} a_{ih}^2} \]  

(15)

where \( r_{ij}^2 \) denotes the observed correlation coefficient for the variables \( x_i \) and \( x_j \); and \( a_{ij}^2 \) represents the partial correlation coefficients. When applying factor analysis techniques we assume the following hypothesis: the correlation coefficients between specific factors for different variables are equal to zero. Therefore, when we have a suitable sample for factor analysis purposes, \( a_{ij}^2 \) will tend to zero. Then KMO’s statistic value will be close to one. Usually values above 0.5 are accepted.

If a positive result is obtained when implementing the aforementioned tests, the process goes on to the second stage. Otherwise, the next step carries the procedure to the third stage.

2) Factor analysis: Risk factors information is condensed into a reduced number of variables by using factor analysis techniques. The starting point for these techniques is a model where a set of observable variables is explained by a set of common factors and a set of specific ones (16):

\[ x_1 = p_{11}F_1 + p_{12}F_2 + \cdots + p_{1j}F_j + \cdots + p_{1m}F_m + e_1 \]

\[ x_2 = p_{21}F_1 + p_{22}F_2 + \cdots + p_{2j}F_j + \cdots + p_{2m}F_m + e_2 \]

\[ \vdots \]

\[ x_i = p_{i1}F_1 + p_{i2}F_2 + \cdots + p_{ij}F_j + \cdots + p_{im}F_m + e_i \]

\[ \vdots \]

\[ x_n = p_{n1}F_1 + p_{n2}F_2 + \cdots + p_{nj}F_j + \cdots + p_{nm}F_m + e_n \]  

(16)

where \( x_i \) (\( i = 1, \ldots, n \)) denotes the \( i \)th observable variable; \( e_i \), represents the specific factor for this \( i \)th variable; \( F_j \) (\( j = 1, \ldots, m \)) denotes the \( j \)th factor, common for all the considered variables; and \( p_{ij} \) represents the factorial charge.

The application of factor analysis techniques presents a drawback: part of the original information (the specificity) is lost. However this procedure
has an important advantage: the resulting multivariate factors are uncorrelated.

(3) ARCH tests: In this stage the factors are analysed individually, in order to determine whether an ARCH scheme is applicable on one. This individual analysis is possible thanks to the multivariate factors generated in the preceding stage: these factors are uncorrelated and, therefore, variance and covariance joint estimation is not required (covariances are theoretically equal to zero). In this way the ARCH factor methodology overcomes the drawback that multivariate ARCH models present.

This stage requires calculating specific ARCH tests in order to determine if factor behaviour can be summarised within an ARCH scheme. For this identification process, we can use the Ljung–Box’s $Q$-statistic and/or the Lagrange multiplier test proposed by Engle (1982).

As Enders (1996) states, if the residuals of the generation process follow an ARCH scheme, the conditional variance for these residuals ($h_t$) will be determined by

$$h_t = E[\varepsilon_t^2 / \Psi_{t-1}] = \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^{p} \beta_i h_{t-i}$$

where variables and parameters are similar to those defined for expression (9). In this situation, squared residuals will be autocorrelated. Therefore, the first way to identify an ARCH scheme is to study this autocorrelation. This can be done by calculating the Ljung–Box’s $Q$-statistic:

$$Q = T(T + 2) \sum_{i=1}^{M} (T - i)^{-1} r_i^2$$

where $r_i$ denotes the sample autocorrelation coefficient for squared residuals, corresponding to the $i$th lag; $T$ is the number of observations; and $M$ is the number of autocorrelation coefficients included in the sum. Under the null hypothesis that every autocorrelation coefficient, for squared residuals, is equal to zero, $Q$ follows a $\chi^2_{M-n}$ distribution ($M - n$: degrees of freedom, where $n$ denotes the number of parameters estimated in the regression model). When rejecting this null hypothesis, we are accepting an ARCH scheme for the residuals.

A second possibility for identifying an ARCH scheme is the Lagrange multiplier test proposed by Engle (1982). This requires the estimation of the regression model and calculation of its residuals. Subsequently, we must regress the squared residuals on a constant and on $n$ lagged values for these residuals:

$$\hat{e}_t^2 = \alpha_0 + \sum_{i=1}^{n} \alpha_i \hat{e}_{t-i}^2.$$  \hspace{1cm} (19)

If there are no ARCH effects, the estimated values of $\alpha_1, \ldots, \alpha_n$ should be zero. Hence, this regression will have little explanatory power so the coefficient of determination will be quite low. With a sample of $T$ residuals, under the null hypothesis of no ARCH errors, the statistic $TR^2$ converges to a $\chi^2$ distribution with $n$ degrees of freedom. If $TR^2$ is sufficiently large, rejection of the null hypothesis that $\alpha_1, \ldots, \alpha_n$ are jointly equal to zero is equivalent to rejecting the null hypothesis of no ARCH errors. On the other hand, if it is sufficiently low, it is possible to conclude that there are no ARCH errors.

If we obtain negative results when implementing the aforementioned identification tests, we must look for alternative methodology for VaR calculation. Otherwise, we can proceed to the fourth stage.

(4) Coefficients estimation: In this stage, ARCH model parameters, for each factor conditional variance, are calculated. As Bera and Higgins (1993) state, the most usual process for estimating ARCH models is the maximum likelihood approach. If we assume, as usual, a normal behaviour for the dependent variable defined in expression (6), the conditional distribution of this variable will be defined by two parameters: mean and variance:

$$y_t / \Psi_{t-1} \sim N(X'_t \beta, h_t)$$

and the log likelihood function will be defined by

$$\log(L) = -\frac{T}{2} \log(2\pi) - 0.5 \sum_{t=1}^{T} \log(h_t) - 0.5 \sum_{t=1}^{T} \frac{\hat{e}_t^2}{h_t}$$  \hspace{1cm} (21)

where $L$ is the likelihood function for a normal distribution, and $T$ is the number of useful observations within the sample considered.
If we want to estimate, for instance, a GARCH (1,1) model, the conditional variance will be defined by
\[ h_t = a_0 + a_1 \varepsilon_t^2 + \beta_1 h_{t-1} \]  
(22)

Thus, in expression (21), if we replace the conditional variance \( h_t \) by its value, defined according to (22), we can establish a log likelihood maximisation process by changing values for three parameters: \( a_0, a_1 \) and \( \beta_1 \). Most of the applied work on ARCH models use the Berndt et al. (1974) algorithm for this maximisation procedure.

(5) Variance forecasting: With the coefficients estimated in the previous stage, forecasts of future factor variance values, using past (historical) data, can be made.

(6) VaR calculation: By applying Eq. (2), future portfolio variance values can be estimated from the forecasted factor variance values (at this point it must be remembered that we are working with null covariances). With these estimations, VaR calculation can be done by applying Eq. (1).

In short, following the six stages outlined above, portfolio VaR can be calculated using ARCH models. Thus, the new ARCH factor methodology overcomes the drawbacks encountered when using multivariate ARCH models.

4. ARCH factor methodology implementation

In this section we apply the ARCH factor methodology to a set of portfolios. We work with 50 foreign exchange risk exposed portfolios. Each one is made up of randomly generated positions in five currencies: United States Dollar, Japanese Yen, Swiss Franc, Australian Dollar and Canadian Dollar. We have adopted the European Currency Unit (ECU) (as a proxy for the Euro) as the currency of reference. We obtained every currency quotation from the daily quotations published by a European Central Bank, from January 1990 to December 1996. We use one risk factor for every currency considered, defined as the daily return on currency rate.

The 90–96 observation period is divided into five overlapping periods: 90–92, 91–93, 92–94, 93–95 and 94–96. Each one is used to make estimations over the following year. Hence, 90–92 currency quotations are used to estimate 1993 VaR; 91–93 is used to estimate 1994 VaR; and so on. Considering the preceding division, ARCH factor methodology is applied as follows:

- **Stages 1 and 2:** We obtained positive results in the risk factors correlation analysis: sampling adequacy statistical tests reject, for all the periods, the correlations equal to zero null hypothesis (Table 1). Moreover, as shown in this table, the number of estimated factors was 2 for every period, and the explained variance cumulative percentage remained over 80% for all of them. Eigenvalue over one was the criterion followed to select the number of factors in every period. The Anderson–Rubin method was used to calculate values for the multivariate factors obtained.
- **Stage 3:** We analyse the factors autocorrelation in the overlapping periods and determine that autocorrelation is not statistically significant in any period for factor 1: \( Q \) Ljung–Box statistic reaches values associated with a critical significance value under 5% (12, 24 and 36 lags

<table>
<thead>
<tr>
<th>Used estimation period</th>
<th>Prediction period</th>
<th>KMO</th>
<th>Bartlett</th>
<th>Number of factors</th>
<th>Cumulative (%)</th>
</tr>
</thead>
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<tr>
<td>90–92</td>
<td>93</td>
<td>0.80055</td>
<td>2703.5529*</td>
<td>2</td>
<td>83.9</td>
</tr>
<tr>
<td>91–93</td>
<td>94</td>
<td>0.80074</td>
<td>2872.4675*</td>
<td>2</td>
<td>85.3</td>
</tr>
<tr>
<td>92–94</td>
<td>95</td>
<td>0.76636</td>
<td>2490.4788*</td>
<td>2</td>
<td>82.6</td>
</tr>
<tr>
<td>93–95</td>
<td>96</td>
<td>0.73869</td>
<td>2184.2839*</td>
<td>2</td>
<td>80.2</td>
</tr>
</tbody>
</table>

KMO: Kaiser–Meyer–Oklin measure of sampling adequacy. KMO > 0.5 positive result for factor analysis.
Bartlett: Bartlett test of sphericity; Null hypothesis (Nh): correlations equal to zero; (*) denotes Nh rejection at a 5% significance level.
Number of factors: eigenvalue over 1.
Cumulative (%): explained variance cumulative percentage.
for Ljung–Box statistic are considered). For the other factor (factor 2), autocorrelation reaches significant values in three of the four considered periods. For them, an autoregressive (AR) model is estimated.

The squared residuals of the AR models and, when these are not estimated, the squared values of the factors are analysed in order to determine their suitability for an ARCH scheme. To be precise, their autocorrelation is analysed through Ljung–Box statistic and Lagrange multiplier test. Table 2 summarises the results obtained.

As seen in Table 2, autocorrelation is statistically significant ($Q$-statistic values), and the Lagrange multiplier test rejects its null hypothesis for both factors in all the periods. Therefore, an ARCH scheme is valid for modelling factor behaviour in all the periods.

- **Stage 4**: We estimate several ARCH ($p$) and GARCH ($p,q$) models for every factor and period. We estimate ARCH ($p$) models up to a lag ($p$) equal to 7, and GARCH ($p,q$) models up to lags ($p$ and $q$) equal to 2. Table 3 summarises the optimal models selected, for every factor and period, using the Akaike information criterion (AIC) and the Schwartz Bayesian criterion (SBC).

Additionally, Table 4 shows the statistically significant coefficients calculated for every model, using the Brendt, Hall, Hall and Hausman algorithm within a maximum likelihood estimation procedure.

- **Stages 5 and 6**: Using the parameters estimated in the previous stage, we make risk factors variance forecasts. Thus, parameters estimated for the 90–92 period are used to make 1993 variance forecasts; those estimated for the 91–93 period are used in 1994; and so on. From these forecasts, portfolio variance is estimated by Eq. (2) (portfolio return sensitivity to factors is estimated by ordinary least squares), considering that covariances are equal to zero. Finally portfolio VaR is calculated by Eq. (1). A holding period of one day and a 99% confidence level ($\phi = 2.33$ assuming normal distribution) are considered.

## Results evaluation

In this section we determine that the VaR calculated in the preceding section is in accordance with the assumed confidence level. Moreover, we...
compare the obtained results with those provided by the widely used J.P. Morgan Riskmetrics methodology. J.P. Morgan Riskmetrics is an exponentially weighted moving average methodology. It uses Eq. (5) to calculate forecasts about risk factors variances, and uses similar equations for covariance forecasts. By using a 0.94 decay factor parameter (see Morgan and Reuters, 1996), and a 3-year historical period, we determine portfolio variance through Eq. (2). Subsequently, we calculate the portfolio VaR by Eq. (1), considering a one day holding period \((\tau = 1)\) and a 0.99 confidence level \((\phi = 2.33)\) assuming normal distribution.

We analyse the difference between the results provided by Riskmetrics and ARCH factor methodologies, by estimating the number of days when the calculated VaR is higher than the real portfolio negative return. We divided this number by the total number of days when the portfolio has had a negative return. By expressing this quotient as a percentage (from now on hedge percentage), we get a measure for the actual confidence level of the calculated risk measure. We repeated the preceding calculation procedure for each of the 50 analysed portfolios.

Table 5 summarises the results obtained. As shown, ARCH factor methodology provides a VaR which is higher than portfolio negative returns, on average, in 99.06\% of the analysed days. This percentage is in accordance with the assumed statistical confidence level (99\%). However, when we estimate VaR using Riskmetrics methodology, we obtain a smaller percentage: only 98.43\%, lower than the expected 99\% (statistical confidence level).

We analyse the difference between the percentages provided by both methodologies, by implementing two statistical tests: the binomial test proposed by McNeil and Frey (2000) and the two sample Kolmogorov–Smirnov test.

Firstly we use the backtesting binomial test proposed by McNeil and Frey (2000) who use the
deviation from the expected number of violations (in accordance with the assumed confidence level), to determine whether different methods calculate the VaR adequately.

As they state, the violation indicator (a violation occurs when the estimated VaR \( (\hat{V}_t) \) is lower than the actual loss \( (APL_{t+1}) \) at time \( t \) is Bernoulli distributed:

\[
I_t = 1_{\{APL_{t+1} > \hat{V}_t\}} \sim \text{Be}(1 - q)
\]

where \( q \) is the assumed confidence level (0.99).

Therefore, the total number of violations is binomially distributed:

\[
\sum_{t \in T} I_t \sim B(T, 1 - q)
\]

where \( T \) denotes the total number of observations (number of days when VaR forecasts have been calculated).

Under the null hypothesis that a method provides a number of violations in accordance with the assumed confidence level \( (1 - q) \), if we count more violations than the expected number \( [(1 - q)T] \), we can perform a one-sided binomial test of the null hypothesis against the alternative that the method systematically underestimates the portfolio VaR. Otherwise, if we count fewer violations than expected, we can perform another onesided binomial test of the null hypothesis against the alternative that the method systematically overestimates the portfolio VaR.

The corresponding binomial probabilities are given in Table 6 alongside the number of violations for each method. A “p-value” less than 0.01 is interpreted as evidence against the null hypothesis.

In 7 out of 50 cases, ARCH factor methodology leads to the rejection of the null hypothesis. This number grows to 10 when Riskmetrics is used. Furthermore, all the Riskmetrics rejections are portfolio loss underestimations, while only four ARCH factor rejections correspond to underestimation, which is a better result.

Secondly, we implement the two-sample Kolmogorov–Smirnov test. We have two samples: one defined by the hedge percentage calculated when applying Riskmetrics methodology, and the other defined by this percentage when calculated through ARCH factor methodology. Kolmogorov–Smirnov statistic tests the null hypothesis of homogeneity between the populations where both samples come from.

We calculate the cumulative distribution function for both samples and determine the extreme value for the distance (in absolute value) between them. This maximum value is 0.62, whereas the listed value for a 5% significance level and for a 50 items sample size is 0.272. Then, we reject the null hypothesis and we can say that there are significant statistical differences when using Riskmetrics or ARCH factor methodology.

As we have stated, on average, hedge percentage is higher when using ARCH factor methodology. Now, we implement another test in order to determine whether this property exists for all the sample items and not only for their average value. If this happens, values for the ARCH factor hedge percentage cumulative distribution function will remain below values for this function when Riskmetrics methodology is used.

Fig. 1 represents both cumulative distribution functions. As shown, those values calculated when Riskmetrics methodology is implemented are higher than those obtained when using ARCH factor methodology, if we do not consider two isolated values. This is a first sign indicating that the aforementioned average property can be extrapolated to the whole sample. However, we calculated a second statistical test, in order to confirm this sign formally.

With this aim, we calculate another Kolmogorov–Smirnov test. The null hypothesis assumes that the cumulative distribution functions are equal; the alternative hypothesis, that one of

<table>
<thead>
<tr>
<th>Methodology</th>
<th>Average</th>
<th>Confidence interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>ARCH factor</td>
<td>0.9906</td>
<td>0.9883 0.9929</td>
</tr>
<tr>
<td>Riskmetrics</td>
<td>0.9843</td>
<td>0.9831 0.9854</td>
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</tbody>
</table>

Average: Average value for the quotient between the number of days when calculated VaR is higher than actual portfolio negative return, and the total number of days when a negative return occurs.

Confidence interval: 95% confidence interval for the preceding average value.
these functions is higher than the other one; or to be precise, that cumulative distribution function values are higher when Riskmetrics methodology is implemented:

\[ H_0 : F(x) = G(x), \quad H_1 : F(x) > G(x), \]  

(25)

where \( F(x) \) and \( G(x) \) represent, respectively, the cumulative distribution functions when Riskmetrics and ARCH factor methodologies are implemented.

Testing the preceding hypothesis implies estimating a new statistic: \( (D_{n,n}^+) \), where \( D_{n,n}^+ \) denotes the supreme value for the difference between the aforementioned cumulative distribution functions:

\[ D_{n,n}^+ = \sup (F(x) - G(x)) \]  

(26)

where \( n \) is the sample size (50 items). By applying Eq. (26), we obtain a value equal to 0.3844 for the statistic. The value which divides the acceptance and rejection areas is defined by

Table 6
Binomial test results: number of violations for every portfolio

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>AF (0.053)</th>
<th>RM (0.200)</th>
<th>Portfolio</th>
<th>AF (0.020)</th>
<th>RM (0.298)</th>
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</thead>
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</tbody>
</table>

AF/RM: ARCH factor/Riskmetrics methodologies.
Number of observations (\( T \)): 904.
Theoretically expected number of violations: \( (1 - q)T = 9 \).
\( p \)-Values for a binomial test are given in brackets.

Fig. 1. Cumulative density functions for hedge percentage.
\[ \frac{\chi^2_{2, \alpha}}{2n} \]  

(27)

where \( \alpha \) denotes the selected significance level.

Expression (27) provides a value equal to 0.0599 for a 5\% significance level. This is lower than that calculated when applying (26), i.e. 0.3884. Therefore, we reject the null hypothesis tested, in the face of the considered alternative hypothesis.

In summary, when we use ARCH factor methodology we get higher values for the hedge percentage than those provided by Riskmetrics. Furthermore, the final values are below the expected confidence level, whereas the values provided by ARCH factor methodology are in accordance with the assumed statistical likelihood level.

6. Concluding remarks

VaR has become a key topic when quantifying portfolio market risk. It is especially useful for financial institutions when calculating the prudential capital requirements that their supervisors have established. Moreover, VaR can be used as a tool to evaluate the performance of portfolio managers: it provides information that can be jointly considered with portfolio return for this purpose.

Research on VaR shows that there are several methods available when calculating the portfolio VaR, each one starting from a different set of hypotheses. Not all the methods, therefore, provide similar outcomes; consequently, research in this field is open to new contributions that can improve existing methods.

In this paper we put forward a new model for quantifying the portfolio VaR: the ARCH factor methodology. It combines ARCH models with factor analysis techniques, using the null correlation property that estimated multivariate factors have. This allows us to use ARCH models to calculate the portfolio variance from risk factor variances. Therefore, the proposed method overcomes the drawback of the excessive number of parameters required for the estimation of multivariate ARCH models, which make these models unworkable for several VaR calculation approaches.

ARCH factor methodology was implemented to calculate the VaR on a set of foreign exchange risk exposed portfolios. The results obtained were compared with those provided by J.P. Morgan’s Riskmetrics method. The latter provides a lower than expected hedge percentage, considering the assumed statistical confidence level. On the contrary, when ARCH factor methodology is used, we reach a hedge percentage in accordance with the assumed confidence level; hence, a more accurate fit is obtained.

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