A fuzzy goal programming approach to portfolio selection

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Abstract

Portfolio selection is a usual multiobjective problem. This paper will try to deal with the optimum portfolio for a private investor, taking into account three criteria: return, risk and liquidity. These objectives, in general, are not crisp from the point of view of the investor, so we will deal with them in fuzzy terms. The problem formulation is a goal programming (G.P.) one, where the goals and the constraints are fuzzy. We will apply a fuzzy G.P. approach to the above problem to obtain a solution. Then, we will offer the investor help in handling the results. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

In the 1950s a well-defined theoretical structure for stock market analysis, the portfolio analysis, started up. Modern portfolio theory is based on the pioneering works of Markowitz (1952, 1959) and Sharpe (1963).

Markowitz’s portfolio optimization model, contrary to its theoretical reputation, has not been used extensively from its original form to construct a large-scale portfolio (Konno and Yamazaki, 1991). The first reason behind this is in the nature of the input required for portfolio analysis. If accurate expectations about future mean returns for each stock and the correlation of returns between each pair of stocks could be obtained, then the Markowitz model would produce optimum portfolios. The problem lies in obtaining accurate expectations of the input needed for this model. This difficulty is particularly acute with respect to the matrix of covariances between securities (Elton and Gruber, 1973). Another one of the most significant reasons for this is the computational difficulty associated with solving a large-scale quadratic programming problem with a dense covariance matrix.

Several authors have tried to alleviate these problems by using various approximation schemes (Sharpe, 1967, 1971; Stone, 1973) to obtain linear problems. In recent years, efforts to transform the quadratic problem into a linear one have been outlined in Konno and Yamazaki (1991). These authors have proposed a measure $L_1$ of risk instead
of an $L_2$ measure of the Markowitz model and then they formulated a linear problem.

The use of the index model enables one to reduce the amount of computation by introducing the notion of “factors” influencing stock prices (Perold, 1984; Sharpe, 1963). Yet, these efforts have been largely discounted because of the popularity of equilibrium models such as the capital asset pricing model (CAPM), developed by Sharpe (1964, 1970) and Lintner (1965), and the multifactor arbitrage pricing theory (APT) formulated by Ross (1976) and extended by Huberman (1982) and Connor (1982), which are computationally less demanding. In a factor model, the random return of each security is a linear combination of a small number of common factors, plus an asset-specific random variable (Connor and Korajczyk, 1995).

Portfolio Management deals with the problem of how to derive technical decision rules for buying, holding and selling both risky and riskless assets over time in consistency with investor’s preferences. For diverse reasons, holdings on some assets may be required to be greater than some minimum or below some allowable maximums. Sometimes these bounds exist for economic reasons, such as when a portfolio holding must be maintained above a minimum to preserve a controlling interest in some firm or general linear constraints are imposed usually to accomplish certain objectives, for example, limiting the exposure of a portfolio to fluctuations in one or more common factors (Perold, 1984). All this brings about the use of linear policy constraints in the portfolio optimization framework, to ensure compliance with upper and lower bounds as derived from economic, managerial and legal considerations (Pogue, 1970).

The coefficients of linear policy constraints should be duly recognized and they are usually imposed by the decision maker. Thus, through policy constraints, some degree of subjective imprecision will be introduced into the system. This conclusion leads to the issue of capturing subjective imprecision mathematically in the problem setting. We will therefore work on the possibility of handling policy coefficients and stipulations as fuzzy numbers. The analysis connects the portfolio problem to fuzzy set theory (Östermark, 1989).

Many decision makers might follow a satisfaction criterion rather than the criterion of optimization an objective function. Goal programming (G.P.) – originally developed by Charnes and Cooper (1961) – is a mathematical programming technique with the capability of handling multiple objectives following a satisfaction criterion rather than an optimization one, that satisfaction criterion leading to the concept of goal. The pioneering applications of G.P. for portfolio selection are due to Lee and Lerro (1973). For further applications see also Lee and Chesser (1980). When attributes and/or goals are in an imprecise environment and they cannot be stated with precision, we work on fuzzy G.P. In this paper we shall discuss the construction and application to Spanish funds of G.P. for portfolio selection that takes into account three criteria: return, risk and liquidity. These objectives, in general, are not crisp from the point of view of the investor, so we will deal with them in fuzzy terms. The problem formulation is a G.P. one, where the goals and the constraints are fuzzy. We will apply a fuzzy G.P. approach (Arenas Parra et al., 1997; Arenas, 1997) to the above problem to obtain a solution. Then, we will offer the investor help in handling the results. Section 2 briefly discusses the theoretical background of the multifactor approaches. In Section 3 we discuss a multiple criteria model selection. In Section 4 we survey application model to Spanish funds. We provide some concluding comments in Section 5.

2. Presentation of the multifactor model

We introduce the multicriteria model using the expected return and risk of portfolio, $p$, according to the multifactor model. Modeling the covariances of intersecurity returns is one of the most important aspects of a portfolio analysis (Connor and Korajczyk, 1995). The most widely used model today is the multifactor approach, where the source of covariation amongst returns is attributed to a few common factors (Markowitz and Perold, 1981). We expose below the portfolio’s expected return, $E(R_p)$, and variance, $\text{Var}(R_p)$, respectively:
\[ E(\hat{R}_p) = \sum_{i=1}^{n} x_i E(\hat{r}_i), \]
\[ \text{Var}(\hat{R}_p) = \sum_{i,j=1}^{n} \sigma_{ij} x_i x_j, \]

where \( n \) denotes the number of assets, \( x_i \) the weight of \( i \)th asset in the portfolio \( p \), \( \hat{r}_i \) the return of asset \( i \)th and \( \sigma_{ij} \) denotes the covariance between the return of the assets \( i \) and \( j \).

Briefly in the multifactor model it is supposed that the return of \( i \)th stock, \( \hat{r}_i \), can be represented as (Perold, 1984)
\[ \hat{r}_i = \alpha_i + \sum_{j=1}^{k} \beta_{ij} \hat{F}_j + \hat{\epsilon}_i, \]

where \( \alpha_i \) and \( \beta_{ij} \) are constants and \( \hat{\epsilon}_i \) (idiosyncratic risk) is an asset-specific random variable with a mean of zero and variance \( \sigma^2_{\hat{\epsilon}_i} \) that measures the variability of stock \( i \) that is not attributable to changes in any factor. The factor \( \hat{F}_j \) is, also, a random variable.

It is assumed that there is no correlation between the independent movements of any two securities in the market, i.e., \( \text{Cov}(\hat{\epsilon}_i, \hat{\epsilon}_j) = 0 \) if \( i \neq j \), i.e., the idiosyncratic risk are assumed not to be correlated with one another. And the factors and idiosyncratic risk are uncorrelated:
\[ \text{Cov}(\hat{F}_j, \hat{\epsilon}_i) = 0 \quad j = 1, \ldots, k; \quad i = 1, \ldots, n. \]

It is straightforward to deduce that the implied covariance matrix, \( C \), has the form
\[ C = D + B'PB, \]

where \( D \) is a diagonal matrix with \( i \)th diagonal entry \( \sigma^2_{\hat{\epsilon}_i} \), \( B \) the \( k \times n \) matrix of coefficients \( \beta_{ij} \) representing asset sensitivity to movements in the factors, called factor betas or factor loadings and \( P \) is the \( k \times k \) covariance matrix of the factors.

If, as is usual in many models, the factors are uncorrelated:
\[ \text{Cov}(\hat{F}_j, \hat{F}_k) = 0 \quad \text{if} \quad j \neq k, \]
then the matrix \( P \) is diagonal.

3. The multiindex model for portfolio selection

Our formulation is a particular case of the general multiindex model presented in Esteve (1994). We have considered – according to Esteve – that the random return of \( i \)th assets \( \hat{r}_i \) can be represented as
\[ \hat{r}_i = \alpha_i + \beta_{r(i)} \hat{g}(i) + \hat{\epsilon}_i, \]

where the index \( \hat{g}(i) \) and the residue, \( \hat{\epsilon}_i \), are random variables with a mean of zero. Expression (6) suggests that randomness of the return of \( i \)th security arises from two random variables: \( \hat{g}(i) \) and \( \hat{\epsilon}_i \). \( \hat{g}(i) \) refers to “the index group \( g(i) \)” and measures the common variability of every asset of group \( g(i) \). The residue \( \hat{\epsilon}_i \) measures randomness of \( \hat{r}_i \) that cannot be explained for its index \( \hat{g}(i) \). The coefficient \( \beta_{r(i)} \) represents the sensitivity of asset \( i \) to movements in the index \( \hat{g}(i) \). Also we suppose that the index \( \hat{g}(i) \) is a linear combination of \( k \) factors according to
\[ \hat{g}(i) = \sum_{j=1}^{k} \beta_{g(i)j} \hat{F}_j + \hat{\epsilon}_{g(i)} \]

Substituting (7) in (6) we obtain
\[ \hat{r}_i = \alpha_i + \sum_{j=1}^{k} \beta_{r(i)j} \hat{F}_j + \beta_{r(i)} \hat{\epsilon}_{g(i)} + \hat{\epsilon}_i. \]

The expected return of \( \hat{r}_i, E(\hat{r}_i) \), is equal to \( \alpha_i \) because
\[ E(\hat{g}(i)) = E(\hat{\epsilon}_i) = 0. \]

The expression (8) exposes the return, \( \hat{r}_i \), can be obtained as expected return plus the sum of three sources of random return, factor return and two idiosyncratic returns: asset-specific random variable, \( \hat{\epsilon}_i \), and group-specific random variable \( \hat{\epsilon}_{g(i)} \). The expected return of the portfolio, \( E(\hat{R}_p) \), with holdings vector \( x = (x_1, \ldots, x_n) \) will be obtained as
\[ E(\hat{R}_p) = \sum_{i=1}^{n} x_i \hat{r}_i. \] (10)

We assume
\[
\begin{align*}
\text{Var} (\hat{F}_k) &= 1, \\
\text{Cov} (\hat{e}_{g(i)}, \hat{e}_{g(j)}) &= 0 \quad \text{if } g(i) \neq g(j), \\
\text{Cov} (\hat{F}_k, \hat{F}_l) &= 0 \quad \text{if } l \neq k, \\
\text{Cov} (\hat{F}_k, \hat{e}_{g(i)}) &= 0, \\
\text{Cov} (\hat{e}_i, \hat{e}_j) &= 0 \quad \text{if } i \neq j, \\
\text{Cov} (\hat{e}_i, \hat{e}_{g(j)}) &= 0,
\end{align*}
\] (11)

where \( k \in \{1, 2, 3\}, \ l \in \{1, 2, 3\} \) and \( i, j \in \{1, 2, \ldots, n\} \). Then the covariance matrix, \( C = (\sigma_{ij})_{1 \leq i, j \leq n} \), can be obtained by applying the following formulae:

\[
\begin{align*}
\sigma_{ij} &= \beta_{g(i)} \beta_{g(j)} (b_{g(i)} b_{g(j)} + b_{g(i)} b_{g(j)} + b_{g(i)} b_{g(j)}), \\
&\quad \text{if } g(i) \neq g(j), \\
\sigma_{ij} &= \beta_{g(i)} \beta_{g(j)} (b_{g(i)} b_{g(j)} + b_{g(i)} b_{g(j)} + b_{g(i)} b_{g(j)}), \\
&\quad + b_{g(i)} \beta_{g(j)} \text{Var} (\hat{e}_{g(i)}) \text{Var} (\hat{e}_{g(j)}) \text{Var} (\hat{e}_i), \\
\sigma_{ii} &= \beta_{g(i)}^2 \left( b_{g(i)}^2 + b_{g(i)}^2 + b_{g(i)}^2 \right) \\
&\quad + \beta_{g(i)}^2 \text{Var} (\hat{e}_{g(i)}) + \text{Var} (\hat{e}_i).
\end{align*}
\] (12)

From (6) we obtain
\[
\text{Var} (\hat{r}_i) = \beta_{g(i)}^2 \text{Var} (\hat{\bar{r}}_{g(i)}) + \text{Var} (\hat{e}_i). \] (13)

Since \( \text{Var} (\hat{\bar{r}}_{g(i)}) = 1 \), then
\[
\text{Var} (\hat{e}_i) = \text{Var} (\hat{r}_i) - \beta_{g(i)}^2. \] (14)

And from (7)
\[
\text{Var} (\hat{\bar{r}}_{g(i)}) = b_{g(i)}^2 \text{Var} (\hat{F}_1) + b_{g(i)}^2 \text{Var} (\hat{F}_2) + b_{g(i)}^2 \text{Var} (\hat{F}_3), \] (15)

as \( \text{Var} (\hat{\bar{r}}_{g(i)}) = \text{Var} (\hat{F}_1) = \text{Var} (\hat{F}_2) = \text{Var} (\hat{F}_3) = 1 \)

we obtain
\[
\text{Var} (\hat{e}_{g(i)}) = 1 - b_{g(i)}^2 - b_{g(i)}^2 - b_{g(i)}^2. \] (16)

4. The multiple criteria model for portfolio selection

We consider the multiple criteria model using three criteria: the expected return of the portfolio; the variance return of the portfolio; the portfolio’s liquidity. Liquidity has been measured as the possibility of converting an investment into cash without any significant loss in value. Other things being equal the investor prefers greater liquidity.

The constraints of the problem are represented by a linear system \( Ax \leq b \), and the non-negativity constraints require all the variables of the model to be greater than, or equal to, zero \( (x \geq 0) \), i.e., short sales are not permitted.

The G.P. is perhaps the most widely used approach in the field of multiple criteria decision making as we can see in the extensive bibliographical surveys by Romero (1991). The major advantage of the G.P. model is its great flexibility which enables the decision maker to easily incorporate numerous variations of constraints and goals. In the literature (Hwang and Masud, 1979; Ignizio, 1979; Romero, 1991), there have been a number of different approaches in the construction of the achievement function (objective function). Thus, in some cases goals have been simply multiplied by a weighting factor, while, in others, distinct priority levels have been defined to be examined sequentially. The problem of portfolio selection can be formulated as the following G.P. model:

Determine \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \) such that:

\[
\begin{align*}
E(\hat{R}_p) &\approx r, \\
\text{Var}(\hat{R}_p) &\approx \sigma, \\
L(p) &\approx l, \quad \text{(17)} \\
Ax &\leq b, \\
x &\geq 0,
\end{align*}
\]

with
\[
L(p) = \sum_{i=1}^{n} l(i)x_i, \quad \text{(18)}
\]

where \( l(i) \) denotes the liquidity of the asset \( i \).

The rigid constraints reflect the availability limitations of resources and correspond to the
constraints in the conventional programming model. The goals represent the objectives set with the right-hand side of each goal consisting of the target value.

The above derivations are based on crisp goals and constraints. Considering that the target values and the parameters in the constraints are usually imposed by the decision maker, some degree of subjective imprecision is included. To study this imprecise nature the fuzzy theory and the possibility theory might be more helpful.

Possibility theory has been proposed by Zadeh (1978) and advanced by Dubois and Prade (1985) where fuzzy variables are associated with possibility distributions. Possibility distributions are represented as normal convex fuzzy sets, such as L-R fuzzy numbers, quadratic and exponential functions. As an application of possibility theory to portfolio analysis, possibility portfolio selection models were initially proposed in paper (Tanaka and Guo, 1999).

A fuzzy number is a fuzzy set in the real line which is a normal \(^1\) and bounded convex (Delgado et al., 1994; Dubois and Prade, 1980; Zadeh, 1978). The fuzzy numbers represent the continuous possibility distributions for fuzzy parameters and hence place a constraint on the possible values the variable may assume. From the definition of a fuzzy number \(\tilde{N} = (n_l, n_r, n_z)\), \(^2\) it is significant to note that the set of level \(z\), \(N_z\), can be represented by the closed interval which depends on the value of \(z\): \(N_z = [n_z, n_z^\delta]\).

The formulation of a G.P. problem, if we assume now that the objectives \((\tilde{Z}_s, s = 1, \ldots, p)\), target values \((\tilde{g}_s, s = 1, \ldots, p)\) and constraints \((\tilde{A}x \leq \tilde{b})\) are given in fuzzy terms, takes the form (Arenas et al., 1999):

Find a decision \(x \in \mathcal{X}(\tilde{A}, \tilde{b}) = \{x \in \mathbb{R}^n / \tilde{A}x \leq \tilde{b}, x \geq 0\}\) such that

\[
\tilde{Z}_s \approx \tilde{g}_s, \quad s = 1, \ldots, p,
\]

where the relation \(\approx\) is a relation between fuzzy numbers which will be read as “approximately equal”. This is not any more an optimization problem as there is no objective to maximize or minimize but we have to find a vector which “approximates to” \(\tilde{g}_s\) which acts as a target.

In order to solve this problem we shall develop a G.P. model based on expected intervals \(^3\) of the fuzzy numbers that define target values (Arenas, 1997; Arenas et al., 1998; Arenas et al., 1999). We have used expected intervals for targets because they can be easily calculated when working with triangular or trapezoidal fuzzy numbers. In this way, we will have an interval G.P. model.

For each \(s\) we try that the expected interval of the fuzzy number which defines the \(s\)th objective function is the nearest possible to the expected interval of the \(s\)th component of the solution. We have to solve the following G.P. problem:

Find an \(x \in \mathcal{X}(\tilde{A}, \tilde{b})\) such that

\[
EI(\tilde{Z}_s) \approx EI(\tilde{g}_s), \quad s = 1, \ldots, p. \tag{19}
\]

The uncertain and/or imprecise nature of the technological matrix and of the vector of resources which define the set of constraints of the model, implies that the feasibility of the decision vector can only be guaranteed by considering the minimum feasible set corresponding to level \(z = 0\), i.e.,

\[
\mathcal{F} = \{x \in \mathbb{R}^n / A^k x \leq b^k, x \geq 0\},
\]

which is the intersection of all feasible sets possible (Arenas et al., 1999). However, if the decision maker was ready to run some risk with respect to feasibility, he could establish the level of tolerance he is ready to assume in every constraint through a parameter \(\delta_i \in [0, 1]\), which may vary from some restrictions to others. With the vector \(\delta = (\delta_1, \ldots, \delta_m)\), the new feasible set with which we shall be working will be

---

\(^1\) A fuzzy set is normal if the supreme of its membership function is equal to 1.

\(^2\) We work with triangular fuzzy numbers.

\(^3\) Heilpern (1992) defines the expected interval of a fuzzy number \(\tilde{N}\), which will be noted \(EI(\tilde{N})\). In terms of \(z\)-cuts the expected interval is

\[
EI(\tilde{N}) = \left[ \int_0^1 n^\delta_z dz \right. \left. - \int_0^1 n^\ell_z dz \right],
\]

where \(n^\delta_z\) and \(n^\ell_z\) are the upper and lower ends, respectively, of the \(z\)-cut of the fuzzy number.
\[(\mathcal{F}_\delta) = \left\{ \begin{array}{ll} \left[ A_i^R - \delta_i (A_i^R - A_i^L) \right] x \leq b_i^L + \delta_i (b_i^R - b_i^L), \\
 x \geq 0, \end{array} \right. \forall i \in I,\]
\[(20)\]

where \( A_{i=0} = [A_i^L, A_i^R] \) and \( b_{i=0} = [b_i^L, b_i^R] \) \( \forall \) \( i \in I \) are the \( \alpha \)-cut, \( \alpha = 0 \), of the coefficients of the technological matrix and of the vector of resources which define the set of constraints of the model. Therefore, the problem to be analyzed will be to find a \( x \in \mathcal{F}_\delta \) such that

\[E(\bar{Z}_s) \approx E(\bar{g}_s), \quad s = 1, \ldots, p. \]
\[(21)\]

We shall take as preferred the least wide interval, i.e., the least imprecise, and the relation \( \approx \) in (21) will be interpreted as “approximately equal and less imprecise”. In order to solve the model we introduce the following definitions and results:

**Definition 4.1.** Given two intervals \( A = [a^L, a^R] \) and \( B = [b^L, b^R] \) of the real straight line, such that the width of the first is greater than the width of the second, we shall define the operation *difference by extremes*, which we shall denote by \( \ominus \), as the interval: \( A \ominus B = [a^L - b^L, a^R - b^R] \).

**Definition 4.2.** Given an interval of the real straight line \( A = [a^L, a^R] \) we shall define the absolute value of such interval in the following way:

\[|A| = \left\{ \begin{array}{ll} |a^L| & \text{if } a^L \geq 0, \\
|a^R| & \text{if } a^L < 0 < a^R, \\
|a^L - a^R| & \text{if } a^L \leq 0. \end{array} \right. \]
\[(22)\]

**Definition 4.3.** Given \( A = [a^L, a^R] \) and \( B = [b^L, b^R] \) with \( b^R - b^L \leq a^R - a^L \), we shall define the distance between both intervals as a new interval obtained as the absolute value of its difference by extremes: \( D_{(A,B)} = |A \ominus B| \).

From the equality between intervals and the definition of distance we have just formulated we have (Arenas, 1997):

**Theorem 4.1.** Given \( A = [a^L, a^R] \) and \( B = [b^L, b^R] \) intervals of the real straight line, it verifies: \( A = B \iff D_{(A,B)} = 0 \).

From this theorem the problem of the relation \( \approx \) between the intervals can be transformed into one of minimizing the distance between them. By applying the definition of distance to the expected intervals, we have that

\[D_s = \left| EI(\bar{g}_s) \ominus EI(\bar{Z}_s) \right| \quad \forall s,
\]

and then

\[E(\bar{Z}_s) = E(\bar{g}_s) \iff D_s = 0 \quad \forall s.
\]

In order to work with the G.P. approach we are going to include into the model the positive (represented by \( p_j \)) and negative (represented by \( n_i \)) deviation variables for each goal. These variables quantify in terms of the extremes of the intervals how far the solution of (21) is from aspiration levels set by the decision maker. The negative deviation variables quantify the underachievement of an objective with respect to its level of aspiration, while the positive ones do the same with respect to the overachievement. Then, we can write:

\[E(\bar{Z}_s)^L + n_s^L - p_s^L = E(\bar{g}_s)^L \quad \forall s,
\]
\[E(\bar{Z}_s)^R + n_s^R - p_s^R = E(\bar{g}_s)^R \quad \forall s,
\]

so substituting in the expression of the total deviation for this goal, we have

\[D_s = ||n_s^L - p_s^L, n_s^R - p_s^R||,
\]
\[(24)\]

and taking into account the definition of absolute value of an interval and the properties of deviation variables, we have the following proposition:

**Proposition 4.2.**

\[D_s = [D_s^L, D_s^R] = \left[ n_s^L + p_s^R, \max(p_s^L, n_s^R) \right].
\]
will consist of determining the non-desired deviation variables. The following propositions will indicate the non-desired variables.

**Proposition 4.3.** If $D^R = 0$, then goal interval coincides with the level of aspiration interval.

Therefore, in order to obtain the equality aimed at, it would be sufficient to minimize the upper extreme of every distance interval. The process of minimizing the extremes of the distance intervals can be approached in different ways, giving rise every one of them to a variant of G.P.; the difference between them lies in the achievement function, remaining identical the working constraints.

The achievement function to be used is defined in terms of the positive and negative deviation variables, it is a linear function of positive coefficients of the upper extremes of the distances and it will be represented as

$$
\min \sum_{s=1}^{p} \alpha_s \max \left(p_s^L, n_s^R\right).
$$

(25)

Let us assume that we want to achieve every goal simultaneously. If we denote by $v_s = \max \left(p_s^L, n_s^R\right)$, we shall have to minimize the weighting sum of the $v_s$, then the problem will be

$$
\min \sum_{s=1}^{p} \alpha_s v_s
$$

subject to

\begin{align*}
& p_s^L \leq v_s, n_s^R \leq v_s, \quad s = 1, \ldots, p, \\
& E(\tilde{Z}_s) + n_s^R - p_s^L = E(\tilde{g}_s), \quad s = 1, \ldots, p, \\
& E(\tilde{Z}_s) + n_s^R - p_s^L = E(\tilde{g}_s), \quad s = 1, \ldots, p, \\
& x \in \mathcal{F}_\beta, \\
& n_s^L - p_s^L \leq n_s^R - p_s^L, \quad s = 1, \ldots, p, \\
& n_s^L \times p_s^L = n_s^R \times p_s^R = 0, \quad s = 1, \ldots, p, \\
& n_s^L \geq 0, p_s^L \geq 0, n_s^R \geq 0, p_s^R \geq 0, \quad s = 1, \ldots, p.
\end{align*}

where symbols with a tilde on top, like $\tilde{r}, \tilde{\sigma}, \tilde{l}, \tilde{\tilde{l}}, \tilde{\tilde{A}}$ and $\tilde{b}$ represent fuzzy parameters whose possibility distributions are given by fuzzy numbers. Then we have two crisp objectives, $E(\tilde{R}_p)$ and $\text{Var}(\tilde{R}_p)$, with fuzzy target values, $\tilde{r}$ and $\tilde{\sigma}$, according with the investor’s desires. The liquidity has been considered as a fuzzy objective because it is completely subjective since it depends on the expert opinion.

Using (19) and (20) we have for each $\delta \in [0, 1]$, we will be to find $x \in \mathcal{F}_\delta$ such that:

$$
\begin{align*}
E(\tilde{R}_p) & \approx E(\tilde{r}), \\
\text{Var}(\tilde{R}_p) & \approx E(\tilde{\sigma}), \\
I(L(p)) & \approx E(\tilde{l}),
\end{align*}
$$

(27)

In order to solve this problem we are going to apply the method developed in Arenas (1997). In the following section we shall discuss an application of the fuzzy G.P. selection model we have studied in this section to Spanish mutual funds.

5. Application to Spanish mutual funds

In this section, we will apply the above fuzzy G.P. portfolio selection model to the data of 132 Spanish mutual funds. The data handled are extended over 23 periods of three months each, from the second quarter of 1991 to the last of 1996.

In order to work with them, the funds have been classified into eight groups:

1-FIAMM national (54 funds);
2-FIAMM international (8 funds);
3-FIM national fixed return (24 funds);
4-FIM national variable return (18 funds);
5-FIM national mixed variable return (9 funds);
6-FIM international fixed return (3 funds);
4-FIM national mixed fixed return (14 funds); 8-FIM international variable return (4 funds); where FIAMM means Monetary Market Assets Fund and FIM means Securities Fund.

We have applied factorial analysis to returns of the assets of each group, producing a factor for each one, named group index. We have \( I_1, I_2, \ldots, I_8 \).

To calculate each \( \beta \)-group in (6) we have made a linear regression of fund return \( \hat{r}_i \) with respect to its index \( I_{\hat{g}(i)} \). Then, as indexes were correlated, a new factorial analysis over the eight indexes corresponding one to each group was made and the following orthogonal factors were obtained as shown in Table 1.

From the previous results, we have decided to work with three factors: \( F_1, F_2 \) and \( F_3 \) which can explain, altogether, about 95.4% of the variance of the eight indexes. For the 23 analyzed periods these are the factors and parameters as shown in Tables 2 and 3.

Once all these were computed, the covariance matrix was obtained from (12), (14) and (16). Then, we can set our second objective, i.e., portfolio variance.

As constraints to our application, we have, as usual, the budget constraint: sum of proportions \( x_i \) invested in each asset \( i \) must be equal to one, \( \sum x_i = 1 \); short-term sales are not permitted, \( x_i \geq 0 \); a fuzzy constraint reflecting the DM’s decision not to have more than about 30% of his portfolio in international FIM funds (groups 7 and 8) and \( \delta = 0.5 \).

With this information we have formulated the particular problem of the portfolio selection we have worked on

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<th>Table 1</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Factorial analysis</strong></td>
</tr>
<tr>
<td>Factor</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>6</td>
</tr>
<tr>
<td>7</td>
</tr>
</tbody>
</table>

*Pct of Var means percentage of explained variance. Cum Pct means cumulated percentage of explained variance.

<table>
<thead>
<tr>
<th>Table 2</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Factors</strong></td>
</tr>
<tr>
<td>( F_1 )</td>
</tr>
<tr>
<td>---</td>
</tr>
<tr>
<td>-0.13925</td>
</tr>
<tr>
<td>-0.28994</td>
</tr>
<tr>
<td>-1.10584</td>
</tr>
<tr>
<td>0.40944</td>
</tr>
<tr>
<td>-0.60693</td>
</tr>
<tr>
<td>-1.99312</td>
</tr>
<tr>
<td>0.24567</td>
</tr>
<tr>
<td>0.94939</td>
</tr>
<tr>
<td>0.29477</td>
</tr>
<tr>
<td>0.64217</td>
</tr>
<tr>
<td>0.69279</td>
</tr>
<tr>
<td>-0.02096</td>
</tr>
<tr>
<td>-1.14979</td>
</tr>
<tr>
<td>-0.81208</td>
</tr>
<tr>
<td>-1.07416</td>
</tr>
<tr>
<td>-1.27424</td>
</tr>
<tr>
<td>0.88483</td>
</tr>
<tr>
<td>0.07672</td>
</tr>
<tr>
<td>0.5962</td>
</tr>
<tr>
<td>0.1894</td>
</tr>
<tr>
<td>0.89908</td>
</tr>
<tr>
<td>-0.1446</td>
</tr>
<tr>
<td>2.73045</td>
</tr>
</tbody>
</table>

*The parameter \( b_{ij} \) on formula (7) was obtained through multiple linear regression of each index over the factors: \( F_1, F_2 \) and \( F_3 \).

\[
\text{minimize} \sum_{r=1}^{3} v_r
\]

subject to

\[
p_t^L \leq v_r, n_r^R \leq v_r, \quad r = 1, 2, 3,
\]

\[
\sum_{i=1}^{132} x_i E(\hat{r}_i) + n_1^L - p_1^L = EI(\bar{r})^L,
\]

\[
\sum_{i=1}^{132} x_i E(\hat{r}_i) + n_1^R - p_1^R = EI(\bar{r})^R,
\]

\[
\sum_{i,j=1}^{132} \sigma_{ij} x_i x_j + n_2^L - p_2^L = EI(\bar{r})^L,
\]

\[
\sum_{i,j=1}^{132} \sigma_{ij} x_i x_j + n_2^R - p_2^R = EI(\bar{r})^R,
\]

\[
\sum_{i=1}^{132} EI(\bar{I}(g(i))^L x_i + n_3^L - p_3^L = EI(\bar{I})^L,
\]

\[
\sum_{i=1}^{132} EI(\bar{I}(g(i))^R x_i + n_3^R - p_3^R = EI(\bar{I})^R.
\]
Table 3
Multiple linear regression*

<table>
<thead>
<tr>
<th>INDEX</th>
<th>$b_{\beta(i)}$</th>
<th>$b_{\beta(i)2}$</th>
<th>$b_{\beta(i)3}$</th>
<th>$R^2$</th>
<th>Sign F</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_1$</td>
<td>-0.009821</td>
<td>0.971857</td>
<td>0.111804</td>
<td>0.96889</td>
<td>0</td>
</tr>
<tr>
<td>$I_2$</td>
<td>0.032313</td>
<td>0.977905</td>
<td>0.137097</td>
<td>0.97953</td>
<td>0</td>
</tr>
<tr>
<td>$I_3$</td>
<td>0.677299</td>
<td>0.660559</td>
<td>0.308313</td>
<td>0.92396</td>
<td>0</td>
</tr>
<tr>
<td>$I_4$</td>
<td>0.965183</td>
<td>-0.018177</td>
<td>0.192824</td>
<td>0.98963</td>
<td>0</td>
</tr>
<tr>
<td>$I_5$</td>
<td>0.944668</td>
<td>0.291175</td>
<td>0.413057</td>
<td>0.98195</td>
<td>0</td>
</tr>
<tr>
<td>$I_6$</td>
<td>0.965178</td>
<td>-0.051155</td>
<td>0.223725</td>
<td>0.97331</td>
<td>0</td>
</tr>
<tr>
<td>$I_7$</td>
<td>0.508524</td>
<td>0.699645</td>
<td>0.727393</td>
<td>0.89418</td>
<td>0</td>
</tr>
<tr>
<td>$I_8$</td>
<td>-0.732952</td>
<td>0.264325</td>
<td>0.862826</td>
<td>0.91939</td>
<td>0</td>
</tr>
</tbody>
</table>

$x_{35} = 0.0140$,  
$x_{56} = 0.0186$,  
$x_{58} = 0.0117$,  
$x_{61} = 0.0234$,  
$x_{67} = 0.0104$,  
$x_{73} = 0.0148$,  
$x_{74} = 0.0389$,  
$x_{80} = 0.0151$,  
$x_{81} = 0.0126$,  
$x_{115} = 0.4368$.

(31)

Almost 50% of the associated portfolio to this solution is fund 115 of the group 6-FIM, which is the most profitable of the funds considered. This is coherent with the result obtained maximizing the return (without taking into account risk and liquidty) which gives a single fund portfolio, just number 115, with profitability equal to 4.3762 and a variance of 55.8316.

The least risk portfolio consists on 9 funds in several proportions. We need to underline, due to their relative importance, funds 43, 46, 58 and 76, that have not significance on the solution, because the minimum variance is 0.1707 far from the target value that is “approximately equal” to 15.

Finally, if we observe that more liquid funds correspond to less profitability, then if the 3rd target value increases the consequential portfolio has less profitability; i.e., if $\bar{l} = (0.5, 0.6, 0.75)$, $\bar{r} = (2.4, 2.8, 3)$ and $\bar{\sigma} = (1.3, 5)$, we obtain the following portfolio:

$E(\bar{R}_p) = 2.7235$,  
$\text{Var}(\bar{R}_p) = 3$,  
$\bar{L}(p) = (0.5982, 0.7380, 0.7380)$,  

and

$x_1 = 0.0136$,  
x_2 = 0.0198,  
x_3 = 0.0151$,  
x_4 = 0.0102,  
x_7 = 0.0113,  
x_8 = 0.0270$.
\[ x_{10} = 0.0127, \quad x_{33} = 0.0287, \quad x_{18} = 0.0167, \]
\[ x_{20} = 0.0120, \quad x_{32} = 0.0121, \quad x_{24} = 0.0254, \]
\[ x_{27} = 0.0180, \quad x_{38} = 0.0219, \quad x_{34} = 0.0160, \]
\[ x_{40} = 0.0105, \quad x_{44} = 0.0121, \quad x_{45} = 0.0216, \]
\[ x_{48} = 0.0214, \quad x_{50} = 0.0136, \quad x_{51} = 0.0120, \]
\[ x_{52} = 0.0189, \quad x_{54} = 0.0192, \quad x_{55} = 0.0189, \]
\[ x_{56} = 0.0199, \quad x_{57} = 0.0100, \quad x_{38} = 0.0203, \]
\[ x_{59} = 0.0118, \quad x_{71} = 0.0826, \quad x_{73} = 0.0182, \]
\[ x_{80} = 0.0178, \quad x_{81} = 0.0166, \quad x_{315} = 0.1805. \]

(33)

We can observe that this portfolio consists of numerous funds of groups 1 and 2 (\( x_1 \) to \( x_{60} \)) which are the most liquid, and the share of fund 115, “the most profitable”, is quite lower than in (31).

6. Conclusions

In this paper we have proposed a method to determine portfolios with fuzzy attributes equal to (in the sense of fuzzy numbers) to fuzzy target values. The described solving method is based on the investor’s preferences and on the G.P. techniques, being the expected intervals of the fuzzy target values the aspiration levels of the G.P. problem. G.P. provides a simultaneous solution for several and conflicting objectives, such as profitability, risk and liquidity.

Taking into account that the \( \alpha \)-cuts of fuzzy number are closed intervals of the real straight line, we might have developed other alternatives based on a finite set of \( \alpha \)-cuts. The more information we introduce in the model, the greater will be the number of variables that we can work with, which obviously increases the complexity of the problem.

We are using the fuzzy sets theory because the liquidity admits a high degree of subjectivity since quantifying the knowledge of an expert. Our aim is to include into our framework linguistic labels, such as “near absolutely liquid”, “sufficient liquid” and “little liquid”. And these natural expressions have a fit representation through fuzzy numbers used in the work.

Our model that is capable of helping the investor to find the efficient portfolio that verifies, as close as possible, his goals. If the investor is risk adverse, the target value of the variance should be small; on the contrary, an adventurous investor would admit higher risk and profitability targets, and lower liquidity targets. Our model can also determine which combination of target values is not feasible, so we offer the investor our help to refine his expectations.

References


Hwang, C.L., Masud, A., 1979. Multiple objective decision making methods and applications. in: Fandel, G., Gal, T.,