Theory and Methodology

Approximate portfolio analysis

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Abstract

This paper presents a portfolio selection model based on the idea of approximation. The model describes a portfolio by its decumulative distribution curve and a preference structure by a family of convex indifference curves. It prescribes the optimal portfolio as the one whose decumulative curve has the highest tangent indifference curve. The model extends the mean–variance model in the sense that it does not restrict the return distributions of assets to be normal. While under the assumption of normality, the model simplifies to the mean–variance model. The model has a measure of risk attitudes that resembles the Arrow–Pratt measure while combining both wealth and probability attitudes. Using this measure, we show that the smaller the curvature of a value function and the larger the curvature of a weighting function, the more risk averse an agent. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

An asset can be characterized as a random variable with a probability distribution over its possible returns. A portfolio is a linear combination of asset variables. By its nature, each portfolio has a continuum of possible returns. It is hardly possible that an investor can grasp a portfolio by attending to all of its possible returns. Instead, he or she has to rely on either approximations or summaries to comprehend it.

In the classic model, the expected return of a portfolio was used as such a summary. The mean–variance model extends the classic model by including one additional parameter, variance, to describe its risk character. It is intuitive that one should choose a portfolio that maximizes mean and minimizes variance. However, the mean–variance model does not approximate very well its claimed normative foundation, the von Neumann–Morgenstern (vN–M) theory. In detail, the following conclusions can be drawn concerning its normative validity (Baron, 1997; Brockett and Kahane, 1992; Chipman, 1972; Liu, 1993):
1. Without restrictions on asset distributions, the mean–variance model fails to be consistent with dominance and the moment-based analysis is inconsistent with the vN–M rationality.
2. When mixtures of asset distributions such as compound lotteries are objects of portfolio choice, the mean–variance model is consistent...
with the vN–M theory if the value function is quadratic, returns are bounded, and unlimited short selling (i.e., borrowing shares from a broker and selling them at the prevailing market price) is not allowed.

3. When linear blends of market assets are not objects of portfolio choice, the mean–variance model is consistent with the vN–M theory if portfolio distributions are normal. However, this consistency is not robust to violations of normality and is not supported theoretically by the law of large numbers. A quadratic value function is not desirable (Mossin, 1973). Normality is at best approximate (Fama, 1965). Therefore, the above three limitations imply that the mean–variance model generally deviates from its expected utility–theoretical foundation. However, the popularity of the mean–variance model is not because of its precision of approximating the vN–M theory but because of its simplicity and the power of its implication as evidenced by the capital asset pricing model of Sharpe and Lintner (Black, 1972).

As an alternative to using summaries, this paper extends the theory of coarse utility of Liu and Shenoy (1995) and presents a portfolio selection model based on the idea of using approximations. The main thrust of the theory of coarse utility is the hypothesis that, in the face of complexity, people may not attend to all the outcomes of a many-outcome choice and inevitably modify the original choice into some approximations. Even though at this moment we do not know how people are using approximations, Liu and Shenoy (1995) describe each many-outcome choice by a set of coarse approximations, each of which is a binary lottery with only one nonzero outcome. They establish a function that measures the utility of coarse approximations based on four rational axioms. They define the utility of a choice to be the utility of the best coarse approximation. Under this new paradigm, Liu and Shenoy (1995) propose and justify a new utility function, called the coarse utility function, for modeling preferences under risk. Like rank- and sign-dependent theory of Luce and Fishburn (1991) and Tversky and Kahneman (1992), the coarse utility theory permits the analysis of phenomena associated with the distortion of probabilities. It provides a better resolution to St. Petersburg paradox than the vN–M theory and its generalizations. Despite its descriptive focus, the coarse utility function has some normative appeal. It is shown to underlie a continuous and weak preference relation. It satisfies the requirement of consistency with the first-order stochastic dominance principle and allows for the violations of the independence axiom, which is the most controversial property of the vN–M theory and the property that has been systematically violated in observed human choice behavior.

This paper applies the coarse utility function to the selection of financial assets as well as their linear blends – portfolios. In this context, it builds a measure of risk attitudes for the coarse utility theory based on the notion of certainty equivalents. The measure resembles the Arrow–Pratt measure while combining both wealth and probability attitudes. It then derives a so-called coarse model for portfolio analysis, which greatly relaxes the restrictions of the mean–variance model on allowable asset distributions and preference structures. The coarse model is consistent with first-degree dominance despite of nonnormality of asset distributions. Under the assumption of normality and risk aversion, the coarse model reduces to the mean–variance model.

An outline of this paper is as follows. To motivate the reader, in Section 2, we briefly introduce the coarse utility theory of Liu and Shenoy (1995). In Section 3, we introduce the notion of indifference curves and represent choice problems graphically using coarse utilities. In Section 4, we discuss the characterization of the investor’s risk attitudes. In Section 5, we propose the coarse model for portfolio selection and its simplification for normal portfolios. In Section 6, we conclude the paper. Sections 3–5 constitute the new contribution of the current work.

2. The coarse utility theory

Suppose a portfolio $X$ has density function $f(x)$ over $\mathbb{R}^+$. Let $p(x) = \int_{-\infty}^{x} f(x) \, dx$. Then, according to Liu and Shenoy (1995), for any return $x$, there is a corresponding coarse approximation $(x, p(x))$. 
Since $0 \leq p(x) \leq 1$, a coarse approximation can be seen as a point in the space $\mathbb{R}^+ \times [0, 1]$. Let $\succsim$ be a binary preference relation for all the coarse approximations in $\mathbb{R}^+ \times [0, 1]$. Liu and Shenoy (1995) define a utility function $V$ for coarse approximations as a mapping on $\mathbb{R}^+ \times [0, 1]$ such that

$$V(x_1, p_1) \succeq V(x_2, p_2)$$

if and only if $(x_1, p_1) \succsim (x_2, p_2)$.

Such a utility function can be seen as a bi-attribute value function, where returns and likelihood are the two attributes to be maximized. Let $\sim$ and $\succ$ be respectively the indifference and the strictly preference relations induced from $\succsim$. Assume that $\succsim$ is a continuous weak order, $\succ$ follows dominance, and $\sim$ follows the cancellation assumption (Roskies, 1965). Liu and Shenoy (1995) show that the utility function $V(x, p)$ is decomposable as follows:

$$V(x, p) = V(x, 1)V(x^*, p)$$

with $V(x, 0) = 0$, $V(x^*, 1) = 1$, and $V(0, p) = 0$ for all $p \in [0, 1]$.

Notice that $V(x, 1)$ measures the value of $x$ under certainty. Therefore, $V(x, 1)$ is an ordinary value function and can be denoted by $v(x)$. $V(x^*, p)$ measures the preference value of probability $p$, with which the most preferred outcome $x^*$ is obtained, and can be denoted by $h(p)$. By order preservation, both $v(x)$ and $h(p)$ are strictly increasing functions with $v(0) = 0$, $v(x^*) = 1$, $h(0) = 0$ and $h(1) = 1$. With $v(x)$ and $h(p)$, $V(x, p)$ can be re-expressed as $V(x, p) = v(x)h(p)$.

For any choice $X$, the coarse approximation $(x, p(x))$ corresponding to outcome $x$ has the utility $V(x, p(x)) = v(x)h(p(x))$. Assume both $v$ and $h$ are continuous. According to Liu and Shenoy (1995), the coarse utility of the portfolio $X$ is the maximum value of coarse approximations:

$$U(X) = \max_{x \in \mathbb{R}^+} v(x)h(p(x)).$$  

According to Kahneman and Tversky (1979), a binary lottery $\{(0, 1-p), (x, p)\}$ with the single nonzero outcome $x$ has utility value $v(x)h(p)$. Therefore, $h(p)$ is a weighting function in the traditional sense. Extensive experiments have reported that a weighting function takes a skew S-shape as shown in Fig. 1 (Kahneman and Tversky, 1979). Tversky and Kahneman (1992) explain the skew S-shaped pattern using the principle of diminishing sensitivity in an ad hoc manner. Here, because $v(x) = V(x, 1)$ and $h(p) = V(x^*, p)$, Liu and Shenoy (1995) apply the principles of diminishing sensitivity and loss aversion to the perception of $V(x, p)$ and provide a unified explanation to the S-shaped pattern of value functions (Kahneman and Tversky, 1979) and the skew S-shaped pattern of weighting functions. For example, $h(p) = V(x^*, p)$ measures the utility value of receiving lottery $\{(0, 1-p), (x^*, p)\}$. When $p$ is close to zero, people value $p$ using the frame of gains of certainty, and when $p$ is large, people value $p$ in the frame of loss of certainty. $h(p)$ is concave in the frame of gains of certainty and convex in the frame of losses of certainty. Thus, the skew S-shaped pattern of weighting functions is consistent with the S-shaped pattern of value functions. Also, it is obvious from $h(p) = V(x^*, p)$ that the size of the extreme returns $x^*$ enters into the valuation of probabilities. The behavior of perceiving $p$ will be different given different absolute size of $x^*$. Some theorists (Hogarth and Einhorn, 1990) attribute this difference to probability $\times$ utility interactions.

The certainty equivalent (CE) to portfolio $X$ is determined by the following equation:

$$v(CE) = \max_{x \in \mathbb{R}^+} v(x)h(p(x)).$$  

![Fig. 1. Observed probability distortion.](image-url)
As shown in Liu and Shenoy (1995), by assessing the CE of a standard lottery, we can assess weights given values or values given weights but not both simultaneously. Because attitudes to outcomes and attitudes to probabilities are not separable, one cannot assess values and weights independently. This difficulty is prevalent in other theories (Tversky and Kahneman, 1992).

Because of using decision weights, the coarse utility function permits the analysis of phenomena associated with the distortion of subjective probability and accommodates some empirical violations of the vN–M theory. For example, it can explain the Allais’ paradox equally well as rank-tations of the vN–M theory. For example, it can explain the Allais’ paradox equally well as rank-tations of the vN–M theory. For example, it can explain the Allais’ paradox equally well as rank-tations of the vN–M theory. For example, it can explain the Allais’ paradox equally well as rank-tations of the vN–M theory.

Utility theory can prescribe more reasonable choices than the expected utility theory and its rank-dependent theories (Liu and Shenoy, 1995) demonstrate, the coarse utility function and the prospect theory of Kahneman and Tversky (1979) are reconciled. Any empirical evidence with such kind of lotteries will equally support both the coarse utility theory and the prospect theory. Besides, as Liu and Shenoy (1995) demonstrate, the coarse utility theory can prescribe more reasonable choices than the expected utility theory and its rank- and sign-dependent generalization for complex choice problems such as the St. Petersburg gamble.

3. Tangency analysis

From this section on, we will extend the coarse utility theory to portfolio selection. First let us re-scale returns as follows so that they are nonnegative:

\[
\text{(the closing price of } t) + (\text{the dividends in } t) - \text{(the closing price of } t-1) \tag{3}
\]

The difference between Eq. (3) and the definition in Markowitz (1959) is a matter of adding or subtracting a constant 1.0. However, we note that Eq. (3) is not as much like a return as a relative future price of an asset. In fact, one often treats Eq. (3) as a “price relative” and its natural log as a “continuously compounded return”. The nonnegativity of returns dictates that we only consider approximations for positive lotteries.

A preference structure is determined by the value function \(v(x)\) and the weighting function \(h(p)\). Intuitively, each point \((x, p)\) in \(\mathbb{R}^+ \times [0, 1]\) gives an agent the satisfaction of degree \(v(x)h(p)\). All \((x, p)\) that make \(v(x)h(p)\) a constant, i.e.,

\[v(x)h(p) = \text{const} (\text{const} \geq 0) \tag{4}\]

will make the agent equally satisfied. Thus, we call the locus of Eq. (4) an indifference curve.

Note that, unlike the term we usually use in economics, an indifference curve here describes a class of indifferent coarse approximations rather than a class of indifferent portfolios.

Recall that both \(v(x)\) and \(h(p)\) are strictly increasing functions. An indifference curve is downward sloping. That is, the return \(x\) and the chance \(p\) are substitutes. To make an investor equally satisfied, the decrease of \(x\) must be compensated by the increase of \(p\) and vice versa.

Theorem 1 shows that indifference curves are also well behaved sometimes.

**Theorem 1.** Assume both \(v(x)\) and \(h(p)\) are second-order differentiable. For any \(c > 0\), \(\{(x, p) | v(x)h(p) \geq c\}\) is convex if either of the following conditions holds:

1. \(v''(x) \leq 0\) and \(h'(p)^2 - \frac{1}{2} h''(p)h(p) \geq 0\);

2. \(v'(x)^2 - \frac{1}{2} v''(x)v(x) \geq 0\) and \(h''(p) \leq 0\).

**Proof.** \(\{(x, p) | v(x)h(p) \geq c\}\) is convex iff \(p\) is an implicit convex function of \(x\), determined by the equation \(V(x)h(p) = c\). Under the smoothness condition, the convexity is equivalent to the condition that \(d^2p/dx^2 \geq 0\). Differentiating the equation \(v(x)h(p) = c\) twice yields

\[
\frac{d^2p}{dx^2} = -\frac{v''(x)(h(p))}{v(x)h''(p)} + \left[\frac{v'(x)}{v(x)}\right]^2 h''(p) \left[2 - \frac{h''(p)h(p)}{h'(p)^2}\right] - \left[\frac{v'(x)}{v(x)}\right]^2 \frac{h''(p)h(p)^2}{h'(p)^3}.
\]
We can easily verify $d^2p/dx^2 \geq 0$ by assuming either condition (1) or (2) and by noting that $v(x)$ and $h(p)$ are both strictly increasing and nonnegative functions. □

Intuitively, the convexity of indifference curves implies that an agent has diminishing marginal rate of substitution between $x$ and $p$. Note that both $v''(x) \leq 0$ and $h'(p)^2 - \frac{1}{2}h''(p)h(p) \geq 0$ are empirically appealing assumptions. The concavity of $v(x)$ implies that the agent has diminishing marginal values. The inequality

$$h'(p)^2 - \frac{1}{2}h''(p)h(p) \geq 0$$

holds for a wide range of functional forms. It is easy to see that Eq. (5) holds when $h(p)$ is concave or linear. In general, the solutions to the differential equation

$$h'(p)^2 - \frac{1}{2}h''(p)h(p) = c(p),$$

where $c(p)$ is nonnegative on $[0, 1]$, satisfy Eq. (5). For example, let $c(p) = 0$. Then, given the boundary condition $h(1) = 1$ and $h'(1) = a > 0$, the solution to Eq. (6) is $h(p) = 1/(1 + a - ap)$. For $a = 2.0$, $3.0$, and $4.0$, the corresponding function curves of $h(p)$ are illustrated in Fig. 2. We can see that Eq. (5) holds even for some convex $h(p)$. If $h(p)$ is skew S-shaped as shown in Fig. 1, $h(p)$ satisfies Eq. (5) iff its convex part is a solution to Eq. (6) for a nonnegative $c(p)$. In fact, the empirical weighting function $h(p)$ in Tversky and Kahneman (1992) satisfies Eq. (5) when $p$ is in $[0.05, 0.95]$, where the estimation of $h(p)$ is reliable.

In Theorem 1, condition (2) is dual to but less interesting than condition (1). As we see in Fig. 1, $h(p)$ is concave only when $p$ is very small. Note that $p(x) = P(X \geq x)$. Small $p$ is associated with large $x$. Condition (2) might be useful when a value function is convex for large returns as long as the convex part of $v(x)$ is the solution to the equation

$$v'(x)^2 - \frac{1}{2}v''(x)v(x) = c(x)$$

for a nonnegative $c(x)$. We shall show shortly that, people are extremely risk seeking if they pick up a very small probability and its associated large return to value a portfolio. Therefore, condition (2) says that the indifference curves of extreme risk lovers are also convex.

In the space $\mathbb{R}^+ \times [0, 1]$, portfolio $X$ with density function $f(x)$ is completely described by its decumulative distribution curve

$$L[X] = \left\{ (x, p) | p = p(x) = \int_x^\infty f(x) \, dx \right\}.$$

For example, assume portfolio $X$ has a uniform density function on $[0, 10]$. Then $L[X]$ is the locus of the function $p = 1 - x/10$. In general, since $p(x)$ is decreasing with $x$, $L[X]$ is a downward sloping curve. In terms of indifference curves and decumulative distribution curves, the utility of a portfolio $X$, according to Eq. (1), is then the optimal value of the problem

$$\max U(X) = v(x)h(p) \quad \text{s.t.} \quad (x, p) \in L[X]. \quad (7)$$

At this point, it is worthwhile comparing the model (7) with the notions of opportunity sets and efficient frontiers in classic portfolio theory. Eq. (7) suggests that we represent each security $X$ by its decumulative distribution curve $L[X]$. The preference value of the security is then measured by the iso-value of the indifference curve tangent to $L[X]$. In the classic portfolio theory, each security $X$ is represented by one point $(\mu_X, \sigma_X)$ in the mean–variance space. The preference value of the security is represented by the iso-value of the indifference curve passing through $(\mu_X, \sigma_X)$. The
collection of all mean–variance points, called an opportunity set, represents a feasible region for a portfolio selection. The minimum variance boundary of an opportunity set is called an efficient frontier. Assume that an investor is risk averse and wants to minimize variance and maximize expected return. The portfolio selection in the classic portfolio theory then becomes as simple as locating the tangent point between the efficient frontier and an indifference curve. Note that the simplicity comes with a price, which is the assumption that every asset has a normal distribution. In Section 5, we will show how we can relax the assumption and make portfolio selections by representing assets by a decision context and each portfolio has density $f(x)$ on $[0, x^\ast]$.

A1. $x^\ast$ is the most preferred return in a given decision context and each portfolio $X$ has density $f(x)$ on $[0, x^\ast]$.

A2. $v(x)$ and $h(p)$ are second-order differentiable.

A3. For each $X$ with density $f(x)$, there is at most one internal point $(x, p)$ on $L[X]$ satisfying

$$v'(x)h(p) - v(x)h'(p)f(x) = 0,$$

where an internal point is a nonterminal point on $L[X]$.

A3 is weaker than the concavity of $L[X]$ to ensure the uniqueness of the tangent point between an indifference curve and $L[X]$.

For example, later we will verify that A3 holds if $X$ is normal or exponential and $v(x)$ and $h(p)$ are linear. Since $v(x)$ and $h(p)$ are increasing functions and $v(0) = 0$ and $h(0) = 0$, A1 ensures that the solution to Eq. (7) is an internal point and satisfies Eq. (8). A3 further assumes the solution to Eq. (8) is unique and so Eq. (8) becomes the sufficient condition for $(x, p)$ to be the solution to Eq. (7).

Therefore, we have Theorem 2, which states that the solution to the problem (7) is the unique tangent point between an indifference curve and a decumulative distribution curve. The coarse utility is the iso-value of the tangent indifference curve.

**Theorem 2.** Under the assumptions A1–A3, $(x, p)$ is the solution to Eq. (7) if and only if it is an internal point on $L[X]$ and satisfies Eq. (8).

**Proof.** We can prove this theorem easily by noting

$$\left.\frac{dv(x)h(p(x))}{dx}\right|_{x=x^*} > 0, \quad \left.\frac{dv}{dx}h(p(x))\right|_{x=x^*} < 0,$$

and by applying the intermediate-value property of $\frac{dv(x)h(p(x))}{dx}$. □

4. Characterization of risk attitudes

In the vN–M theory, Arrow (1971) shows the equivalence between risk aversion and the concavity of a value function. This equivalence does not sustain in more general theories because of additional dimension of attitudes to probability preferences. In rank-dependent utility theories, when a value function is linear, Yaari (1987) shows that an agent is pessimistic if and only if she always overweightes probabilities, and that an agent is optimistic if and only if she always underweights probabilities. If a value function is assumed to be concave, Quiggin (1993) proves that an agent is risk averse if she always overweightes probabilities. In general, Chew et al. (1987) establish that a rank-dependent utility model follows second order stochastic dominance if and only if both a value function and a weighting function are concave. In this section, we characterize risk attitudes in the theory of coarse utility.
Since the value of the function $v(x)h(p)$ at the tangent point is the utility of a portfolio, the tangent point describes an $(x, p)$-equivalent to the portfolio. We can regard the tangent point as an investor’s viewpoint from which he or she sees the portfolio. Therefore, we naturally anticipate that the position of a tangent point carries information about an investor’s risk attitudes. Given the same portfolio, suppose investors 1 and 2, respectively pick $(20\%, 0.95)$ and $(80\%, 0.2)$ as their $(x, p)$-equivalents. We could easily conjecture that investor 1 is more risk averse than investor 2 because investor 1 looks at the portfolio from the perspective of a low return but a high probability while investor 2 looks at it from the perspective of a high return but a low probability. Therefore, along a decumulative distribution curve, we expect that the further left is the $(x, p)$-equivalent, the more risk averse is the investor. For any two investors, the one with the up-left tangent point is more conservative and risk averse than the one with the bottom-right tangent point. For example, as shown in Fig. 3, the CE for investor 1 is CE1 and for investor 2 is CE2. Observe that CE1 < CE2. Thus, according to the theoretical concept of risk attitudes, investor 1 is more risk averse than investor 2. It is consistent with our previous intuitive analysis. In general, we have the following theorem.

**Theorem 4.** Assume at any point $(x, p)$ in $(0, x’) \times (0, 1)$, the slope of the indifference curve for investor 1 is different from that for investor 2. For any portfolio $X$, under assumptions A1–A3, if the tangent point on $L[X]$ for investor 1 is left to that for investor 2, then the certainty equivalent to $X$ for investor 1 is smaller than that for investor 2.

**Proof.** Let $(x_1, p(x_1))$ and $(x_2, p(x_2))$ be the tangent points for investors 1 and 2, respectively. By assumption, $x_1 < x_2$ and $p(x_1) > p(x_2)$. Let ID1 and ID2 be the tangent indifference curves for investors 1 and 2, respectively. Let CE1 and CE2 denote the certainty equivalents to $X$ for investors 1 and 2, respectively. Suppose CE1 ≥ CE2. Then, at the horizon $p=1$, ID1 is right to ID2. ID2 cannot pass any point on or below $(x_1, p(x_1))$ because otherwise it is not the highest tangent indifference curve. Therefore, at the horizon $p=p(x_1)$, ID1 is left to ID2. Similarly, ID1 cannot pass any point on or below $(x_2, p(x_2))$. At the horizon $p=p(x_2)$, ID1 is right to ID2. Therefore, ID2 crosses ID1 twice at A and B from its right to its left as shown in Fig. 4. Because of continuity, the preference value $v_2(x)h_2(p)$ for investor 2 will change smoothly. For any point $(x, p)$ on ID1, which is below A, there must exist another point $(x+\Delta x, p+\Delta p)$, which is also on ID1 but below

![Fig. 3. Comparison of risk attitudes.](image-url)
A, we will have their first-order Taylor series yields at (h, p) = 0.

Replacing \( v_2(x + \Delta x) \) and \( h_2(p + \Delta p) \) above by their first-order Taylor series yields

\[
v_2(x + \Delta x)h_2(p) + v_2(x)h_2'(p)\Delta x + v_2(x)h_2''(p)\Delta p = 0.
\]

Along ID \(_1\) gradually moving \((x, p)\) away from A, we will have \((x + \Delta x, p + \Delta p)\) moving down away from B. Correspondingly, we have \( \Delta x \to 0 \). Therefore, \( \frac{dp}{dx} = -\frac{v_2'(x)h_2(p)}{v_2(x)h_2'(p)} \). Note that, by the construction, \( \frac{dp}{dx} \) is the slope of ID \(_1\) at \((x, p)\) while \(-\frac{v_2'(x)h_2(p)}{v_2(x)h_2'(p)}\) can be verified to be the slope of the indifference curve at \((x, p)\) for investor 2. Therefore, at \((x, p)\), the slopes of the indifference curves for investors 1 and 2 are the same. However, since \((x, p)\) must be in the region \((0, x') \times (0, 1)\), by assumption, this cannot happen.

Given functions \( v(x) \) and \( h(p) \), we define the measure of risk attitudes at \((x, p)\) as follows:

\[
r(x, p) = \frac{v'(x)h(p)}{v(x)h'(p)}.
\]

The larger the \( r(x, p) \) value, the more risk seeking an investor, and vice versa. Note that in the vN–M theory, the Arrow–Pratt measure of risk attitudes is \(-\frac{v''(x)}{v'(x)}\) (Arrow, 1971). There are several differences between the risk measure in the coarse utility theory and in the vN–M theory.

First, in the vN–M theory, utility is linear with probabilities. Its risk measure does not include the characteristics of \( h(p) \). Second, the risk measure in the coarse utility theory uses relative marginal values \( v'(x)/v(x) \) and relative marginal distortions \( h'(p)/h(p) \) while that in the vN–M theory uses the curvature of a value function. Shortly we will relate these two notions.

**Theorem 5.** Assume investors 1 and 2 have risk attitudes respectively denoted by \( r_1(x, p) \) and \( r_2(x, p) \). Under assumptions A1–A3, given any portfolio, the tangent point for investor 1 is left to that for investor 2 iff \( r_1(x, p) < r_2(x, p) \) at the two tangent points.

**Proof.** Assume investor \( i \) has value function \( v_i(x) \) and weighting function \( h_i(p) \) \( (i = 1, 2) \). Assume portfolio \( X \) has density function \( f(x) \).

**Sufficiency:** Assume \((x, p)\) is the tangent point for investor 1. The tangency condition (8) implies \( r_1(x, p) = f(x) \). Thus,

\[
r_2(x, p) = \frac{v_2'(x)h_2(p)}{v_2(x)h_2'(p)} > f(x) = r_1(x, p).
\]

Therefore, according to Theorem 3, investor 2 has the tangent point right to \((x, p)\). Similarly, we can prove that investor 1 has the tangent point left to \((x, p)\) if \((x, p)\) is the tangent point for investor 2.

**Necessity:** Assume \((x_1, p_1)\) and \((x_2, p_2)\) are respectively the tangent points for the investors 1 and 2. Assume \( x_1 < x_2 \) and \( p_1 > p_2 \). According to Theorem 3 and the tangency condition (8),

\[
\frac{h_1(p_1)}{v_1(x_1)} \frac{v_1'(x_1)}{h_1'(p_1)} f(x_1) = 0,
\]

\[
\frac{h_2(p_1)}{v_2(x_1)} \frac{v_2'(x_1)}{h_2'(p_1)} f(x_1) > 0.
\]

Therefore, we have

\[
\frac{h_1(p_1)}{v_1(x_1)} \frac{v_1'(x_1)}{h_1'(p_1)} < f(x_1) = \frac{h_2(p_1)}{v_2(x_1)} \frac{v_2'(x_1)}{h_2'(p_1)}.
\]

Similarly, we can prove \( r_1(x, p) < r_2(x, p) \) at \((x_2, p_2)\).
attitudes. Neither \( v'(x)/v(x) \) nor \( h'(p)/h(p) \) alone can reflect the whole picture of an investor’s risk attitudes.

**Theorem 6.** Assume investors 1 and 2 have respectively value functions \( v_1(x) \) and \( v_2(x) \) and weighting functions \( h_1(p) \) and \( h_2(p) \). Under assumptions A1 and A2, if at any point \((x, p)\),

\[
\frac{v_1''(x)}{v_1'(x)} < \frac{v_2''(x)}{v_2'(x)} \quad \text{and} \quad \frac{h_1'(p)}{h_1(p)} > \frac{h_2'(p)}{h_2(p)},
\]

then \( r_1(x, p) < r_2(x, p) \).

**Proof.** We want to prove that, for any \( x > 0 \), and \( p > 0 \), the following inequalities hold:

\[
\frac{v_1'(0) + v_1''(0)x}{v_1'(x)} < \frac{v_2'(0) + v_2''(0)x}{v_2'(x)}. \tag{11}
\]

Note that only need to show the first inequality of Eq. (11) because the second one can be proved similarly. First we show that Eq. (11) holds for any small \( x \). Replace \( v_i(x) \) and \( v_i'(x) \) \( (i = 1, 2) \) in Eq. (11) by their first-order Taylor expansions at \( x = 0 \). Then Eq. (11) becomes

\[
\frac{v_1'(0) + v_1''(0)x}{v_1'(x)} < \frac{v_2'(0) + v_2''(0)x}{v_2'(x)}. \tag{12}
\]

It is easy to see that Eq. (12) holds under condition (10). Now we show that, if Eq. (11) holds for \( x \) and \( p \), it also holds for \( x + \Delta x \) and \( p + \Delta p \) when \( \Delta x \) and \( \Delta p \) are small positive numbers, i.e.,

\[
\frac{v_1'(x + \Delta x)}{v_1(x + \Delta x)} < \frac{v_2'(x + \Delta x)}{v_2(x + \Delta x)}. \tag{13}
\]

Let \( x = \varepsilon \) be a small positive number at which Eq. (11) holds. Because of the uniform continuity of \( v_i'(x) \) and \( v_i''(x) \) \( (i = 1, 2) \) on \([x, x']\), it is straightforward to prove, according to Eq. (10), that there exists a small positive number \( \delta \) such that, for any \( x \in [x, x'] \), and any \( \xi_1, \xi_2, \eta_1, \eta_2 \in [x, x + \delta] \), the following inequality holds:

\[
\frac{v_1''(\xi_1)}{v_1'(\eta_1)} < \frac{v_2''(\xi_2)}{v_2'(\eta_2)}. \tag{14}
\]

Let \( \Delta x \) be a positive number such that \( \Delta x < \delta \), and

\[
\frac{1}{\Delta x} > \max_{x \in [x', x]} \left\{ \frac{|v_1''(x)|}{|v_1'(x)|}, \frac{|v_2''(x)|}{|v_2'(x)|} \right\}.
\]

Let

\[
v_i'(x + \Delta x) = v_i'(x) + v_i''(\xi) \Delta x \quad (i = 1, 2),
\]

\[
v_i(x + \Delta x) = v_i(x) + v_i'(\eta) \Delta x \quad (i = 1, 2).
\]

Assume Eq. (11) holds for \( x \), i.e.,

\[
0 < \frac{v_1'(x)v_2(x)}{v_1(x)v_2'(x)} < 1. \tag{15}
\]

According to the construction of \( \Delta x \) and inequality (14), we have the following:

\[
0 < 1 + \frac{v_1''(\xi_1)}{v_1'(\eta_1)} \Delta x < 1 + \frac{v_2''(\xi_2)}{v_2'(\eta_2)} \Delta x.
\]

Multiplying the above inequality by Eq. (15) yields

\[
v_1'(x)v_2(x) \left[ 1 + \frac{v_1''(\xi_1)}{v_1'(\eta_1)} \Delta x \right] < v_2'(x)v_1(x) \left[ 1 + \frac{v_2''(\xi_2)}{v_2'(\eta_2)} \Delta x \right]. \tag{16}
\]

Also, according to Eq. (14), the following holds:

\[
v_1''(\xi_1)v_2'(\eta_2)(\Delta x)^2 < v_2''(\xi_2)v_1'(\eta_1)(\Delta x)^2.
\]

Adding the above inequality to Eq. (16) yields

\[
\frac{v_1'(x) + v_1''(\xi_1) \Delta x}{v_1(x) + v_1'(\eta_1) \Delta x} < \frac{v_2'(x) + v_2''(\xi_2) \Delta x}{v_2(x) + v_2'(\eta_2) \Delta x}.
\]

Therefore, Eq. (13) is proved. Because the length of \( \Delta x \) is fixed, after finite steps of induction from Eq. (11) to Eq. (13), we can show Eq. (1) holds for any \( x > 0 \).

Note that when \( x = p = 0 \), Eq. (11) approaches to equalities when \( x, p \to 0 \) by l’Hospital’s rule. Therefore, according to Eq. (11), at any point \((x, p)\) in \((0, x^*) \times (0, 1)\),

\[
r_1(x, p) = \frac{v_1'(x)h_1(p)}{v_1(x)h_1'(p)} < \frac{v_2'(x)h_2(p)}{v_2(x)h_2'(p)} = r_2(x, p).
\]

Therefore, investor 1 is more risk averse than investor 2. \( \square \)
Theorem 6 links the measure of risk attitudes in Eq. (9) to the curvature of \(v(x)\) and \(h(p)\) and shows the consistency between \(r(x, p)\) and the Arrow–Pratt risk measure of risk aversion \((-v''(x)/v'(x))\).

The following corollary can be used to define absolute risk attitudes, from which it seems that the concavity of \(v(x)\) and the convexity of \(h(p)\) are consistent dual behavior. As a matter of fact, they are both observed most frequently in experiments as we noted before.

**Corollary 1.** Under assumptions A1 and A2, an agent with concave \(v(x)\) and convex \(h(p)\) has smaller \(r(x, p)\) than the agent with linear \(v(x)\) and \(h(p)\). An agent with convex \(v(x)\) and concave \(h(p)\) has larger \(r(x, p)\) than the agent with linear \(v(x)\) and \(h(p)\).

In summary, this section has established the four different characterizations of risk attitudes, whose logical interplay is sketched by the following diagram:

- curvature \(\Rightarrow r(x, p)\)
- location of tangency \(\iff\)
- certainty equivalent \(\Rightarrow\)

**5. Portfolio selection**

Indifference curves are defined independently of portfolios. Also, a decumulative distribution curve exists independently of subjective preferences. Therefore, the representation of a portfolio and the representation of an investor's preferences are separated. Assume the homogeneous expectation. Different investors will perceive the same distribution for the same portfolio in spite of their differences in preference structures. Similarly, one investor can use the same set of indifference curves in evaluating different portfolios.

The separation implies that the same set of indifference curves can not only determine the utility of a portfolio but also pick up the optimal portfolio among several alternatives. Indifference curves are well behaved. The higher an indifference curve, the higher \(v(x)h(p)\) value. Therefore, for any two portfolios \(X\) and \(Y\), if \(L[Y]\) is tangent to a higher indifference curve than \(L[X]\), then \(Y\) has a higher

utility value and so is better than \(X\). In general, for any set of portfolios, the one with the highest tangent indifference curve is optimal. This property makes the coarse utility function useful for portfolio selection. For example, given indifference curves IC\(_i\) and two portfolios \(X\) and \(Y\) shown in Fig. 5, \(L[Y]\) is tangent to IC\(_1\) and \(L[X]\) is tangent to IC\(_0\). Since IC\(_1\) is above IC\(_0\), \(Y\) has the higher utility than \(X\). Hence, \(Y\) is selected while \(X\) is rejected. If many portfolios are presented, we simply pick up the one with the highest tangent point.

Given any two portfolios, different investors may have different ranking orders. For example, between \(X\) and \(Y\) shown in Fig. 6, investor 1 prefers \(Y\) to \(X\) while investor 2 prefers \(X\) to \(Y\). As shown in Fig. 6, relatively to \(X\), \(Y\) does the better job in giving returns with large probabilities. On the other hand, \(X\) does the better job in giving high returns. According to Theorem 5, investor 1 is more risk averse and prefers a relatively low return but with a high probability. The portfolio \(Y\) fits the taste of investor 1. On the other hand, investor 2 likes high returns even with low chances. Portfolio \(X\) panders to this taste.

Given any finite number of marketable assets, we can construct infinite number of linear blends. Therefore, we will have infinite number of decumulative curves on \(\mathbb{R}^+ \times [0, 1]\). The totality of all these curves constitute an opportunity set for the whole security market, which is called the market.
opportunity set. Assume that information is toll-
free and simultaneously available to all investors
and that all investors hold the same expectation
about the return of each asset. Then the market
opportunity set is objective and independent of
individual preference structures. All investors in
the market will face the same market opportunity
set. Optimal portfolios are then selected by ma-
nipulating their indifference curves. Since the locus
of any portfolio is bounded above, there exists an
upper boundary for the market opportunity set.
This boundary is called the efficient frontier,
whose meaning is carried by Theorem 7.

Theorem 7. A portfolio is optimal only if its
opportunity set passes through or is tangent to the
efficient frontier.

Proof. Let \( X \) be optimal. Then \( L[X] \) is tangent to
the highest indifference curve. Suppose the tangent
point is \((x, p)\). Then \((x, p)\) must be on the efficient
frontier. Otherwise, there exists a point \((x', p')\) such
that \( x' > x \) and \( p' > p \) and at least one inequality is
strict. Therefore \( v(x')h(p') > v(x)h(p) \). Suppose
\((x', p')\) is on \( L[Y] \) for some \( Y \) in the market
opportunity set. Then
\[
U(Y) \geq v(x')h(p') > v(x)h(p) = U(X).
\]
That is, \( X \) is not optimal. A contradiction results. \( \Box \)

According to Theorem 7, the efficient frontier is
the combination of \((x, p)\) attainable in the market
which maximizes the \( v(x)h(p) \) for any investors. To
find an optimal portfolio in the market, the effi-
cient frontier carries the sufficient information
about all market portfolios. For any investor, the
indifference curve tangent to the efficient frontier
determines the highest level of satisfaction that can
be reached in the market. Any portfolio whose
locus passes through the tangent point on the
efficient frontier is optimal. For example, as
shown in Fig. 7, the bold curve is the efficient
frontier. The tangent point is \( A \). Both \( X \) and \( Z \) are
optimal.

Theorem 8. If \( X \) is first-degree dominant over \( Y \) then
\( U(X) \geq U(Y) \). If for every value function \( v(x) \),
\( U(X) \geq U(Y) \), then \( X \) is first-degree stochastically
dominant over \( Y \).

Proof. A rigorous proof can be found in Liu and
Shenoy (1995). Here we sketch an intuitive one.
Assume the portfolios \( X \) and \( Y \) have respectively
the density functions \( f_X(x) \) and \( f_Y(y) \).

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Proof. A rigorous proof can be found in Liu and
Shenoy (1995). Here we sketch an intuitive one.
Assume the portfolios \( X \) and \( Y \) have respectively
the density functions \( f_X(x) \) and \( f_Y(y) \).
Sufficiency: $X$ is dominant over $Y$. Then for any $x \in (0, \infty), \int_0^x f_X(x) \, dx \leq \int_0^y f_Y(y) \, dy$, which is equivalent to $P\{X > x\} \geq P\{Y \geq x\}$. In other words, $L[X]$ is always above $L[Y]$. Therefore, for any preference structures, $X$ is more preferred than $Y$.

Necessity: For all $v(x)$, $U(X) \geq U(Y)$. It implies that $L[X]$ always has a tangent point which is right-above the tangent point on $L[Y]$. The only possibility is that $L[X]$ always locates above $L[Y]$, which is shown to be equivalent to first-order dominance of $X$ over $Y$. Theorem 8 implies that the coarse model of portfolio selection is consistent with the dominance principle. However, in general, the dominance of $X$ over $Y$ implies $U(X) \succeq U(Y)$ rather than $U(X) > U(Y)$. Indeed, there are situations, where $X$ weakly stochastically dominates $Y$ but $U(X) = U(Y)$. (See Liu and Shenoy (1995) for an example.) The coarse utility function shows its weakness in this respect. One can construct other similar examples. Liu and Shenoy (1995) defend that the coarse utility function approximates the utility of a lottery. Thus, $U(X) = U(Y)$ should be interpreted as $U(Y)$, and this is not inconsistent with our empirical findings. Theorem 8 indicates that the coarse utility function does not admit the violation of dominance but is sometimes incapable of identifying preferences among weakly differentiated lotteries. Note that such incapability exists in any approximate theory because by nature it does not use all available information. The mean–variance or general moment-based analysis admits the violation of dominance when preference structures and/or distributions are not restricted. In this respect, the coarse model has a normative improvement.

With the aid of Theorem 7, the statement of Theorem 8 can be strengthened. Given a portfolio $X$, if for every $(x, p)$ on $L[X]$, there exists an $(x', p')$ on the efficient frontier such that $x \leq x'$, $p \leq p'$, and at least one inequality is strict, then we say $X$ is dominated by the efficient frontier. Note that if $X$ is strictly dominated by other portfolios, $X$ must also be dominated by the efficient frontier. The converse is not generally true. Therefore, the dominance of the efficient frontier is weaker than first-degree dominance.

**Theorem 9.** $X$ is dominated by the efficient frontier if there exist a portfolio $Y$ such that $U(Y) > U(X)$ for every $v(x)$ and $h(p)$.

**Proof.** Necessity results directly from Theorem 7. Now suppose $(x, p) \in L[X]$. Suppose there exists no $(x', p')$ in the market opportunity set such that $x' \geq x$, $p' \geq p$, and at least one inequality is strict. Then there is no portfolio whose locus passing through the up-right area of $(x, p)$. Now we adjust $v(x)$ and $h(p)$ such that the dashed lines in Fig. 8 form an indifference curve. Then for this kind of $v(x)$ and $h(p)$, there is no $Y$ such that $U(Y) > U(X)$. This is a contradiction.

To compare with the performance of the mean–variance model, we now explore the special feature of normal portfolios and their extended family of stable portfolios. A distribution is called stable if the logarithm of its characteristic function is represented by the formula $i \mu t - c|t|^\alpha$, where $\mu$ is a real location parameter, $\alpha$ is the characteristic exponent of a stable distribution or kurtosis parameter with $0 < \alpha < 2$, and $c$ is the scale parameter with $c \geq 0$. The stable class includes normal ($\alpha = 2$), Cauchy ($\alpha = 1$), and one-point distributions ($c = 0$) as its special family members. According to Gnedenko

![Fig. 8. An counter-example of nondominance.](image-url)
and Kolmogorov (1968, pp. 171), a stable distribution with exponent $0 < \alpha < 2$ has only finite absolute moments of order $\delta$ ($0 < \delta < \alpha$). A normal distribution has mean $\mu$ and variance $2\epsilon$, a stable distribution with $1 < \alpha < 2$ has mean $\mu$ but infinite variance, and a distribution with $0 < \alpha \leq 1$ has neither mean nor variance.

The stable class is the only class that is invariant under addition, i.e., the distribution of sums of independent, identically distributed, stable variables is itself stable and has the same parameter $\alpha$. This property makes the stable class useful to model the behavior of stock returns. For example, Kendall (1953) found that weekly stock returns seem to be approximately normally distributed. Mandelbrot (1963) observed and later Fama (1965) confirmed that stock returns are better described by stable distributions with $1 < \alpha < 2$.

**Theorem 10.** Assume $X$ is stable with $1 < \alpha \leq 2$. If $v(x)$ and $h(p)$ are linear, then the tangent point is unique and lies on the concave part of $L[X]$.

**Proof.** Since $v(0) = 0$, $h(0) = 0$, and $h(1) = 1$, let us assume $v(x) = ax$ and $h(p) = p$. Let $X$ have density function $f(x)$ and mean $\mu$, $f(x)$ is symmetric around $x = \mu$. For any $x \geq \mu$, its symmetric point relative to $x = \mu$ is $2\mu - x$. Because $X$ is nonnegative, the following inequality can be observed using the graph of $f(x)$:

$$xf(x) > \int_0^{2\mu-x} f(x) \, dx = \int_0^\infty f(x) \, dx.$$

Thus, at $x \geq \mu$,

$$\frac{d[v(x)h(p)]}{dx} \bigg|_{p=p(x)} = \frac{a}{\gamma} \int_0^{\infty} f(x) \, dx - axf(x) < 0.$$

Note that the decumulative curve $\int_x^\infty f(x) \, dx$ is concave when $x < \mu$ and convex when $x > \mu$. Thus, the tangency condition (8) will only hold on the concave part of $L[X]$. According to Theorem 1, indifference curves are convex. Thus, the point $(x, p)$ satisfying Eq. (8) is unique and must be the solution to problem (7). □

**Corollary 2.** Assume $X$ is stable with $1 < \alpha \leq 2$. Under assumptions A1 and A2, if $v(x)$ is concave and $h(p)$ is convex but satisfies Eq. (5), then the unique tangent point lies on the concave part of $L[X]$.

**Proof.** Let $X$ have density function $f(x)$ and mean $\mu$. According to Corollary 1 and Theorem 10, for any $x \geq \mu$, $r(x, p(x)) < f(x)$. Therefore, the tangency condition (3) only holds on the concave part of $L[X]$. Since $h(p)$ satisfies Eq. (5), all the indifference curves are convex. Thus, the tangent point is unique. □

Theorem 11 states that, as in the vN–M theory, under the assumption of normality, the coarse model of portfolio analysis can be simplified as the mean–variance analysis.

**Theorem 11.** Assume $v(x)$ is concave or linear and $h(p)$ is linear or convex but satisfies Eq. (5). Under assumptions A1 and A2, all normal portfolios can be ranked by their means and variances in a consistent way with the coarse utility theory. Indifference curves in the $\mu–\sigma$ plane exist and have the following characteristics:

1. When $\sigma = 0$, the slope of indifference curves is zero;
2. When $\sigma > 0$, indifference curves are upward-sloping and convex.

**Proof.** For every normal portfolio $X$, its coarse utility is

$$U(X) = \max_{x \in (0, \infty)} v(x)h[p(x, \mu, \sigma)]$$

$$= \max_{y \in (-\infty, \infty)} v(\mu + \sigma y)h[p(y, 0, 1)],$$

where

$$y = \frac{x - \mu}{\sigma}$$

and

$$p(x, \mu, \sigma) = \int_0^\infty \frac{1}{\sqrt{2\pi} \sigma} \exp \left\{ -\frac{1}{2} \left( \frac{x - \mu}{\sigma} \right)^2 \right\} \, dx.$$

Let $g(y) = h[p(y, 0, 1)]$. Assume at $y^*$, $v(\mu + \sigma y)g(y)$ reaches the maximum. Then at $y^*$,

$$\sigma v'(\mu + \sigma y)g(y) + v(\mu + \sigma y)g'(y) = 0.$$

(17)
Now, assume that \( \mu \) and \( \sigma \) change to \( \mu + d\mu \) and \( \sigma + d\sigma \) and correspondingly \( y^* \) changes to \( y^* + dy \). Assume \((\mu, \sigma)\) and \((\mu + d\mu, \sigma + d\sigma)\) are on the same indifference curve in the \( \mu-\sigma \) plane. Then
\[
v(\mu + \sigma y^*)g(y^*)
= v[\mu + d\mu + (\sigma + d\sigma)(y^* + dy)]g(y^* + dy)
= v(\mu + \sigma y^*)[g(y^*) + g'(y^*)dy + \cdots]
+ v'(\mu + \sigma y^*)[d\mu + \sigma dy + d\sigma dy]
+ y^* d\sigma[g(y^*) + g'(y^*) dy + \cdots]
= v(\mu + \sigma y^*)g(y^*) + v(\mu + \sigma y^*)g'(y^*) dy
+ v'(\mu + \sigma y^*)g(y^*)(d\mu + \sigma dy + y^* d\sigma).
\]

Plugging Eq. (17) into the above equality yields
\[
d\mu + y^* d\sigma = 0.\]

When \( \sigma = 0 \), the portfolio has the certain return \( \mu \) and \( y^* = 0 \). Thus, \( d\mu d\sigma = 0 \). When \( \sigma > 0 \), according to Theorem 10 and Corollary 2, \( y^* < 0 \). Thus, \( d\mu d\sigma = -y^* > 0 \). Thus, in the \( \mu-\sigma \) plane, indifference curves are upward sloping (see Fig. 9).

For every \( x \), the concavity of \( v(x) \) implies that, for any \( \lambda \) such that \( 0 < \lambda < 1 \),
\[
v(\lambda \mu_1 + (1 - \lambda)\mu_2 + (\lambda \sigma_1 + (1 - \lambda)\sigma_2)y)g(y)
= v(\lambda (\mu_1 + \sigma_1 y) + (1 - \lambda)(\mu_2 + \sigma_2 y))g(y)
> [v(\lambda (\mu_1 + \sigma_1 y) + (1 - \lambda)(\mu_2 + \sigma_2 y))g(y]
= \lambda v(\mu_1 + \sigma_1 y)g(y) + (1 - \lambda)v(\mu_2 + \sigma_2 y)g(y).
\]

Thus,
\[
\max_{x \in (0, \infty)} v(x) h[p(x, \lambda \mu_1 + (1 - \lambda)\mu_2, \lambda \sigma_1 + (1 - \lambda)\sigma_2)]
> \lambda \max_{x \in (0, \infty)} v(x) h[p(x, \mu_1, \sigma_1)]
+ (1 - \lambda) \max_{x \in (0, \infty)} v(x) h[p(x, \mu_2, \sigma_2)].
\]

Therefore, in the \( \mu-\sigma \) plane, \((\lambda \mu_1 + (1 - \lambda)\mu_2, \lambda \sigma_1 + (1 - \lambda)\sigma_2)\) is on an indifference curve with a higher coarse utility value than \((\mu_1, \sigma_1)\) and \((\mu_2, \sigma_2)\). In other words, indifference curves are convex as shown in Fig. 9. \( \Box \)

Theorem 11 implies that the foundation of the mean–variance model can also be built on the theory of coarse utility. It reconciles the coarse utility theory and the vN–M theory in prescribing normal portfolios. For any two normal portfolios \( X \) and \( Y \), if \( X \) is mean–variance dominant over \( Y \), i.e., \((\mu_X, -\sigma_X) \geq (\mu_Y, -\sigma_Y)\), then for a concave \( v(x) \) and a convex \( h(p) \), \( X \) is preferred to \( Y \) in accordance with maximizing coarse utility. This property renders the coarse utility theory empirically testable.

6. Conclusion

In this paper we first reviewed the theory of coarse utility of Liu and Shenoy (1995) in the special context of selecting financial assets. We then propose the notion of indifference curves to graphically represent coarse utilities. We showed that an indifference curve is convex when the value function \( v(x) \) and the weighting function \( h(p) \) satisfy certain reasonable conditions.

This paper made an important contribution in building a measure of risk attitudes for the coarse utility theory based on the notion of certainty equivalents. Using this measure, we showed that the smaller the curvature of \( v(x) \) and the larger the curvature of \( h(p) \), the more risk averse an agent is. Assume that linear \( v(x) \) and \( h(p) \) correspond to risk neutrality. Then an agent is risk averse if \( v(x) \) is concave and \( h(p) \) is convex, and risk seeking if \( v(x) \) is convex and \( h(p) \) is concave.

We finally derive a graphical model, called the coarse model, for portfolio analysis from the general theory of coarse utility. In the coarse model,
we represent a portfolio by its decumulative distribution and a preference structure by a family of convex indifference curves. We judge a portfolio to be optimal if its decumulative distribution curve has the highest tangent indifference curve. The coarse model is consistent with a weaker form of the first-degree dominance without regard to the assumption of the normality of asset distributions. Therefore, it overcomes some limitations of the mean–variance analysis. Under the assumption of normality and risk aversion, we prove that the coarse model reduces to the mean–variance model. It implies that the coarse model is a nontrivial generalization of the mean–variance model. It also implies that the coarse utility theory can be also a foundation of the modern portfolio analysis, in addition to the expected utility theory.

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