A possibilistic approach to selecting portfolios with highest utility score

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Abstract

The mean–variance methodology for the portfolio selection problem, originally proposed by Markowitz, has been one of the most important research fields in modern finance. In this paper we will assume that: (i) each investor can assign a welfare, or utility, score to competing investment portfolios based on the expected return and risk of the portfolios; and (ii) the rates of return on securities are modelled by possibility distributions rather than probability distributions. We will present an algorithm of complexity $O(n^3)$ for finding an exact optimal solution (in the sense of utility scores) to the $n$-asset portfolio selection problem under possibility distributions. © 2001 Elsevier Science B.V. All rights reserved.

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1. A utility function for ranking portfolios

The mean–variance methodology for the portfolio selection problem, originally proposed by Markowitz [4], has been one of the most important research fields in modern finance theory [7]. The key principle of the mean–variance model is to use the expected return of a portfolio as the investment return and to use the variance of the expected returns of the portfolio as the investment risk.

Following [1] we shall assume that each investor can assign a welfare, or utility, score to competing investment portfolios based on the expected return and risk of those portfolios. The utility score may be viewed as a means of ranking portfolios. Higher utility values are assigned to portfolios with more attractive risk–return profiles. One reasonable function that is commonly employed by financial theorists assigns a risky portfolio $P$ with a risky rate of return $r_P$, an expected rate of return $E(r_P)$ and a variance of the rate of return $\sigma^2(r_P)$, the following utility score [1]:

$$U(P) = E(r_P) - 0.005 \times A \times \sigma^2(r_P),$$

where $A$ is an index of the investor’s risk aversion ($A \approx 2.46$ for an average investor in the USA). The factor of 0.005 is a scaling convention that allows us to express the expected return and standard deviation in Eq. (1) as percentages rather than decimals. Eq. (1) is consistent with the notion that utility is enhanced by high expected returns and diminished by high risk.

Because we can compare utility values to the rate offered on risk-free investments when choosing between a risky portfolio and a safe one, we may

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interpret a portfolio’s utility value as its certainty equivalent rate of return to an investor.

That is, the certainty equivalent rate of return on a portfolio is the rate that risk-free investments would need to offer with certainty to be considered as equally attractive as the risky portfolio. Now, we can say that a portfolio is desirable only if its certainty equivalent return exceeds that of the risk-free alternative. In the mean–variance context, an optimal portfolio selection can be formulated as the following quadratic mathematical programming problem:

\[
U \left( \sum_{i=1}^{n} r_{i}x_{i} \right) = E \left( \sum_{i=1}^{n} r_{i}x_{i} \right) - 0.005 \times A \times \sigma^{2} \\
\times \left( \sum_{i=1}^{n} r_{i}x_{i} \right) \to \max
\]

subject to: \(x_{1} + \cdots + x_{n} = 1, x_{i} \geq 0, \quad i = 1, \ldots, n\),

where \(n\) is the number of available securities, \(x_{i}\) is the proportion invested in security (or asset) \(i\), and \(r_{i}\) denotes the risky rate of return on security \(i\), \(i = 1, \ldots, n\). Denoting the rate of return on the risk-free asset by \(r_{f}\), a portfolio is desirable for the investor if and only if

\[
U \left( \sum_{i=1}^{n} r_{i}x_{i} \right) > r_{f}.
\]

In this paper we will assume that, the rates of return on securities are modelled by possibility distributions rather than probability distributions. That is, the rate of return on the \(i\)th security will be represented by a fuzzy number \(r_{i}\), and \(r_{i}(t)\), \(t \in \mathbb{R}\), will be interpreted as the degree of possibility of the statement that ‘\(t\) will be the rate of return on the \(i\)th security’. In our method, we will consider only trapezoidal possibility distributions, but our method can easily be generalized to the case of possibility distributions of type LR.

In standard portfolio models uncertainty is equated with randomness, which actually combines both objectively observable and testable random events with subjective judgments of the decision maker into probability assessments. A purist on theory would accept the use of probability theory to deal with observable random events, but would frown upon the transformation of subjective judgments to probabilities.

The use of probabilities has another major drawback: the probabilities give an image of precision which is unmerited—we have found cases where the assignment of probabilities is based on very rough, subjective estimates and then the subsequent calculations are carried out with a precision of two decimal points. This shows that the routine use of probabilities is not a good choice. The actual meaning of the results of an analysis may be totally unclear—or results with serious errors may be accepted at face value.

In standard portfolio theory the decision maker assigns utility values to consequences, which are the results of combinations of actions and random events. The choice of utility theory, which builds on a decision maker’s relative preferences for artificial lotteries, is a way to anchor portfolio choices in the von Neumann–Morgenstern axiomatic utility theory. In practical applications the use of utility theory has proved to be problematic (which should be more serious than having axiomatic problems): (i) utility measures cannot be validated inter-subjectively, (ii) the consistency of utility measures cannot be validated across events or contexts for the same subject, (iii) utility measures show discontinuities in empirical tests (as shown by Tversky [5]), which should not happen with rational decision makers if the axiomatic foundation is correct, and (iv) utility measures are artificial and thus hard to use on an intuitive basis.

As the combination of probability assessments with utility theory has these well-known limitations we have explored the use of possibility theory as a substituting conceptual framework.

Let us introduce some definitions, which we shall need in the following section. A fuzzy number \(A\) is called trapezoidal with tolerance interval \([a, b]\), left width \(\alpha\) and right width \(\beta\) if its membership function has the following form:

\[
A(t) = \begin{cases} 
1 - \frac{a - t}{\alpha} & \text{if } a - \alpha \leq t \leq a, \\
1 & \text{if } a \leq t \leq b, \\
1 - \frac{t - b}{\beta} & \text{if } a \leq t \leq b + \beta, \\
0 & \text{otherwise}
\end{cases}
\]

and we use the notation \(A = (a, b, \alpha, \beta)\). It can easily be shown that

\[
[A]^\gamma = [a - (1 - \gamma)\alpha, b + (1 - \gamma)\beta], \quad \forall \gamma \in [0, 1],
\]

where \([A]^\gamma\) denotes the \(\gamma\)-level set of \(A\) (Fig. 1).
Let $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ and $[B]^\gamma = [b_1(\gamma), b_2(\gamma)]$ be fuzzy numbers and let $\lambda \in \mathbb{R}$ be a real number. Using the extension principle we can verify the following rules for addition and scalar multiplication of fuzzy numbers:

$$[A + B]^\gamma = [a_1(\gamma) + b_1(\gamma), a_2(\gamma) + b_2(\gamma)],$$

$$[\lambda A]^\gamma = \lambda [A]^\gamma.$$ 

Let $A \in \mathcal{F}$ be a fuzzy number with $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$, $\gamma \in [0, 1]$. In [2] we introduced the (crisp) possibilistic mean (or expected) value and variance of $A$ as

$$E(A) = \int_0^1 \gamma(a_1(\gamma) + a_2(\gamma)) \, d\gamma,$$

$$\sigma^2(A) = \frac{1}{2} \int_0^1 \gamma(a_2(\gamma) - a_1(\gamma))^2 \, d\gamma.$$

It is easy to see that if $A = (a, b, x, \beta)$ is a trapezoidal fuzzy number then

$$E(A) = \int_0^1 \gamma[a - (1 - \gamma)x + b + (1 - \gamma)\beta] \, d\gamma$$

$$= \frac{a + b}{2} + \frac{\beta - x}{6}$$

and

$$\sigma^2(A) = \frac{(b - a)^2}{4} + \frac{(b - a)(x + \beta)}{6} + \frac{(x + \beta)^2}{24}$$

$$= \left[\frac{b - a}{2} + \frac{x + \beta}{6}\right]^2 + \frac{(x + \beta)^2}{72}.$$

2. A possibilistic approach to portfolio selection problem

Watada [6] proposed a fuzzy portfolio selection model where he used fuzzy numbers to represent the decision maker’s aspiration levels for the expected rate of return and a certain degree of risk. Inuiuchi and Tanino [3] introduced a novel possibilistic programming approach to the portfolio selection problem: their approach, which prefers a distributive investment solution, is based on the minimax regret criterion (the regret which the decision maker is ready to undertake).

In many important cases, it might be easier to estimate the possibility distributions of rates of return on securities, rather than the corresponding probability distributions. Consider now the portfolio selection problem with possibility distributions

$$U \left( \sum_{i=1}^n r_i x_i \right) = E \left( \sum_{i=1}^n r_i x_i \right) - 0.005 \times A \times \sigma^2$$

$$\times \left( \sum_{i=1}^n r_i x_i \right) \rightarrow \max$$

s.t. \$x_1 + \cdots + x_n = 1, x_i \geq 0, \quad i = 1, \ldots, n\$. 

where $r_i = (a_i, b_i, x_i, \beta_i), \ i = 1, \ldots, n$ are fuzzy numbers of trapezoidal form. It is easy to compute that

$$E \left( \sum_{i=1}^n r_i x_i \right) = \sum_{i=1}^n \frac{1}{2} \left[ a_i + b_i + \frac{1}{3}(\beta_i - x_i) \right] x_i$$

and

$$\sigma^2 \left( \sum_{i=1}^n r_i x_i \right) = \left( \sum_{i=1}^n \frac{1}{2} \left[ b_i - a_i + \frac{1}{3}(x_i + \beta_i) \right] x_i \right)^2$$

$$+ \frac{1}{72} \left[ \sum_{i=1}^n (x_i + \beta_i) x_i \right]^2.$$

Introducing the notations

$$u_i = \frac{1}{2} \left[ a_i + b_i + \frac{1}{3}(\beta_i - x_i) \right],$$

$$v_i = \frac{\sqrt{0.005A}}{2} \left[ b_i - a_i + \frac{1}{3}(x_i + \beta_i) \right],$$

$$w_i = \frac{\sqrt{0.005A}}{\sqrt{72}} (x_i + \beta_i)$$
we shall represent the $i$th asset by a triplet $(v_i, w_i, u_i)$, where $u_i$ denotes its possibilistic expected value, and $v_i^2 + w_i^2$ is its possibilistic variance multiplied by the constant $0.005 \times A$. We will also assume that there are at least three distinguishable assets, with the meaning that if two assets have the same expected value and variance then they are considered indistinguishable (or identical in the framework of mean-variance analysis).

That is, we assume that $u_i \neq u_j$ or $v_i^2 + w_i^2 \neq v_j^2 + w_j^2$ for $i \neq j$.

Then we will state the possibilistic portfolio selection problem (3) as

$$\langle u, x \rangle - \langle v, x \rangle^2 - \langle w, x \rangle^2 \rightarrow \max,$$

s.t. $\{x_1 + \cdots + x_n = 1, x_i \geq 0, i = 1, \ldots, n\}.$$

(4)

The convex hull of $\{(v_i, w_i, u_i): i = 1, \ldots, n\}$, denoted by $T$, and defined by

$$T = \text{conv}\{(v_i, w_i, u_i): i = 1, \ldots, n\} = \left\{ \left( \sum_{i=1}^{n} v_i x_i, \sum_{i=1}^{n} w_i x_i, \sum_{i=1}^{n} u_i x_i \right): \sum_{i=1}^{n} x_i = 1, \right.$$

$$x_i \geq 0, i = 1, \ldots, n \left. \right\}$$

is a convex polytope in $\mathbb{R}^3$. Then (4) turns into the following three-dimensional non-linear programming problem:

$$-(v_0^2 + w_0^2 - u_0) \rightarrow \max,$$

s.t. $(v_0, w_0, u_0) \in T,$

or, equivalently,

$$f(v_0, w_0, u_0) := v_0^2 + w_0^2 - u_0 \rightarrow \min,$$

s.t. $(v_0, w_0, u_0) \in T,$

(5)

where $T$ is a compact and convex subset of $\mathbb{R}^3$, and the implicit function

$$g_c(v_0, w_0) := v_0^2 + w_0^2 - c$$

is strictly convex for any $c \in \mathbb{R}$. This means that any optimal solution to (5) must be on the boundary of $T$.

We will now present an algorithm for finding an optimal solution to (4) on the boundary of $T$. Note that, $T$ is a compact and convex polyhedron of $\mathbb{R}^3$ and that any optimal solution to (5) must be on the boundary of $T$, which imply that any optimal solution can be obtained as a convex combination of at most 3 extreme points of $T$. In the algorithm by lifting the non-negativity conditions for investment proportions we shall calculate: (i) the (exact) solutions to all conceivable 3-asset problems with non-colinear assets, (ii) the (exact) solutions to all conceivable 2-assets problems with distinguishable assets, and (iii) the utility value of each asset. Then we compare the utility values of all feasible solutions (i.e. solutions with non-negative weights) and portfolios with the highest utility value will be chosen as optimal solutions to the portfolio selection problem (5). Our algorithm will require $O(n^3)$ steps, where $n$ is the number of available securities.

Consider three assets $(v_i, w_i, u_i), i = 1, 2, 3$, which are not colinear: $\mathbb{R}(x_1, x_2, x_3) \in \mathbb{R}^3, (x_1, x_2, x_3) \neq 0$, such that

$$x_1 \left[ v_1 \atop w_1 \atop u_1 \right] + x_2 \left[ v_2 \atop w_2 \atop u_2 \right] - (x_1 + x_2) \left[ v_3 \atop w_3 \atop u_3 \right] = 0.$$

Then the 3-asset optimal portfolio selection problem with not-necessarily non-negative weights reads

$$(v_1 x_1 + v_2 x_2 + v_3 x_3)^2 + (w_1 x_1 + w_2 x_2 + w_3 x_3)^2$$

$$- (u_1 x_1 + u_2 x_2 + u_3 x_3) \rightarrow \min$$

s.t. $x_1 + x_2 + x_3 = 1$.

(6)

Let us denote

$$L(x, \lambda) = (v_1 x_1 + v_2 x_2 + v_3 x_3)^2$$

$$+ (w_1 x_1 + w_2 x_2 + w_3 x_3)^2$$

$$- (u_1 x_1 + u_2 x_2 + u_3 x_3)$$

$$+ \lambda(x_1 + x_2 + x_3 - 1),$$

(7)
the Lagrange function of the constrained optimization problem (6). The Kuhn–Tucker necessity conditions are

\[
2v_1(v_1x_1 + v_2x_2 + v_3x_3) + 2w_1(w_1x_1 + w_2x_2 + w_3x_3) - u_1 + \lambda = 0,
\]

\[
2v_2(v_1x_1 + v_2x_2 + v_3x_3) + 2w_2(w_1x_1 + w_2x_2 + w_3x_3) - u_2 + \lambda = 0,
\]

\[
2v_3(v_1x_1 + v_2x_2 + v_3x_3) + 2w_3(w_1x_1 + w_2x_2 + w_3x_3) - u_3 + \lambda = 0,
\]

\[
x_1 + x_2 + x_3 = 1,
\]

which leads us to the following linear equality system:

\[
\begin{bmatrix}
q_1^2 + r_1^2 & q_1q_2 + r_1r_2 \\
q_1q_2 + r_1r_2 & q_2^2 + r_2^2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= \begin{bmatrix} 1/2(u_1 - u_3) - q_1v_3 - r_1w_3 \\ 1/2(u_2 - u_3) - q_2v_3 - r_2w_3 \end{bmatrix},
\]

where we used the notations \(q_1 = v_1 - v_3, q_2 = v_2 - v_3, r_1 = w_1 - w_3\) and \(r_2 = w_2 - w_3\).

Now we prove that if \((v_i, w_i, u_i), i = 1, 2, 3\), are not colinear then Eq. (8) has a unique solution. Suppose that the solution to Eq. (8) is not unique, i.e.

\[
det\begin{bmatrix}
q_1^2 + r_1^2 & q_1q_2 + r_1r_2 \\
q_1q_2 + r_1r_2 & q_2^2 + r_2^2
\end{bmatrix} = 0.
\]

That is

\[
det\begin{bmatrix}
q_1^2 + r_1^2 & q_1q_2 + r_1r_2 \\
q_1q_2 + r_1r_2 & q_2^2 + r_2^2
\end{bmatrix} = (q_1^2 + r_1^2)(q_2^2 + r_2^2) - (q_1q_2 + r_1r_2)^2
\]

\[
= (q_1r_2 - q_2r_1)^2 = \left(\det\begin{bmatrix} q_1 & r_1 \\
q_2 & r_2 \end{bmatrix}\right)^2 = 0.
\]

Thus, the rows of

\[
\begin{pmatrix}
q_1 & r_1 \\
q_2 & r_2
\end{pmatrix}
\]

are not linearly independent: \(\exists (x_1, x_2) \neq 0\) such that

\[
x_1[q_1, r_1] + x_2[q_2, r_2] = 0 \iff x_1[v_1 - v_3, w_1 - w_3] + x_2[v_2 - v_3, w_2 - w_3] = 0.
\]

We find that Eq. (8) turns into

\[
(q_2^2 + r_2^2)\begin{bmatrix}
q_2^2 & -x_1x_2 \\
x_1x_2 & x_1^2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= \begin{bmatrix} 1/2x_1(u_1 - u_3) + x_2(q_2v_3 + r_2v_3) \\ 1/2x_1(u_2 - u_3) - x_1(q_2v_3 + r_2v_3) \end{bmatrix}.
\]

Multiplying both sides by \([x_1, x_2]\) we get that \(u_1, u_2\) and \(u_3\) have to satisfy the equation

\[
x_1^2[1/2x_1(u_1 - u_3) - 1/2x_1(u_2 - u_3)] = 0.
\]

If \(x_1 \neq 0\), then we obtain \(x_1(u_1 - u_3) + x_2(u_2 - u_3) = 0\), and from Eq. (9) it follows that

\[
x_1\begin{bmatrix} v_1 \\
w_1 \\
u_1 \end{bmatrix} + x_2\begin{bmatrix} v_2 \\
w_2 \\
u_2 \end{bmatrix} - (x_1 + x_2)\begin{bmatrix} v_3 \\
w_3 \\
u_3 \end{bmatrix} = 0,
\]

i.e. \((v_i, w_i, u_i), i = 1, 2, 3\), were colinear.

If \(x_1 = 0\), then \(x_2 \neq 0\), and from Eq. (9) it follows that \(q_2 = r_2 = 0\). Now we find that Eq. (8) turns into

\[
\begin{bmatrix}
q_1^2 + r_1^2 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= \begin{bmatrix} 1/2(u_1 - u_3) - q_1v_3 - r_1w_3 \\ 1/2(u_2 - u_3) \end{bmatrix}.
\]

Multiplying both sides by \([0, 1]\), we obtain

\[
1/2(u_2 - u_3) = 0.
\]

We find that

\[
v_2 - v_3 = w_2 - w_3 = u_2 - u_3 = 0,
\]

which means that \((v_i, w_i, u_i), i = 1, 2, 3\), were colinear. Which ends the proof.

Using the general inversion formula

\[
\begin{bmatrix} t_1 & t_2 \\
t_3 & t_4 \end{bmatrix}^{-1} = \frac{1}{t_1t_4 - t_2t_3}\begin{bmatrix} t_4 & -t_2 \\
-t_3 & t_1 \end{bmatrix},
\]


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we find that the optimal solution to (8) is
\[
\begin{bmatrix}
x_1^2 \\
x_2^2
\end{bmatrix} = \frac{1}{(q_1 r_2 - q_2 r_1)^2} \times \begin{bmatrix}
q_2^2 + r_2^2 & -(q_1 q_2 + r_1 r_2) \\
-(q_1 q_2 + r_1 r_2) & q_1^2 + r_1^2
\end{bmatrix} \times \begin{bmatrix}
1/2(u_1 - u_3) - q_1 v_3 - r_1 v_3 \\
1/2(u_2 - u_3) - q_2 v_3 - r_2 v_3
\end{bmatrix}.
\]
(10)

We will now show that
\[x^* = (x_1^*, x_2^*, 1 - x_1^* - x_2^*)\]

satisfies the Kuhn–Tucker sufficiency condition, i.e. \(L''(x, \lambda)\) is a positive definite matrix at \(x = x^*\) in the subset defined by
\[
\{ y = (y_1, y_2, y_3) \in \mathbb{R}^3: y_1 + y_2 + y_3 = 0 \}.
\]

Really, from (7) we get
\[
M := \frac{1}{2} L''(x^*, \lambda) = \begin{bmatrix}
v_1^2 + w_1^2 & v_1 v_2 + w_1 w_2 & v_1 v_3 + w_1 w_3 \\
v_1 v_2 + w_1 w_2 & v_2^2 + w_2^2 & v_2 v_3 + w_2 w_3 \\
v_1 v_3 + w_1 w_3 & v_2 v_3 + w_2 w_3 & v_3^2 + w_3^2
\end{bmatrix} = \begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix}^T \begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix}^T + \begin{bmatrix}
w_1 \\
w_2 \\
w_3
\end{bmatrix}^T \begin{bmatrix}
w_1 \\
w_2 \\
w_3
\end{bmatrix},
\]

and, therefore, the inequality
\[
y^T My = (v_1 y_1 + v_2 y_2 + v_3 y_3)^2 + (w_1 y_1 + w_2 y_2 + w_3 y_3)^2 \geq 0 \quad (11)
\]
holds for any \(y \in \mathbb{R}^3\). So \(M\) is a positive semidefinite matrix. If \(y^T My = 0\) for some \(y = (y_1, y_2, y_3) \neq 0, y_1 + y_2 + y_3 = 0\), then from (11) we find
\[
v_1 y_1 + v_2 y_2 + v_3 y_3 = 0, \quad w_1 y_1 + w_2 y_2 + w_3 y_3 = 0
\]
and we would get that
\[
\det \begin{bmatrix}
v_1 & v_2 & v_3 \\
w_1 & w_2 & w_3 \\
1 & 1 & 1
\end{bmatrix} = \det \begin{bmatrix}
q_1 & q_2 \\
r_1 & r_2
\end{bmatrix} = \det \begin{bmatrix}
q_1 & r_1 \\
q_2 & r_2
\end{bmatrix} = 0,
\]
which would lead us to a contradiction with the non-colinearity condition. So \(L''\) is positive definite. Thus \(x^*\) is the unique optimal solution to (6) and \(x^*\) is an optimal solution to (4) (with \(n = 3\)) if \(x_1^* > 0, x_2^* > 0\) and \(x_3^* > 0\) (the Kuhn–Tucker regularity condition). The optimal value of (6) will be denoted by \(U_x\).

Consider now a 2-asset problem with two assets, say \((v_1, w_1, u_1)\) and \((v_2, w_2, u_2)\), such that \((v_1, w_1, u_1) \neq (v_2, w_2, u_2)\):
\[
\begin{align*}
&v_1 x_1 + v_2 x_2)^2 + (w_1 x_1 + w_2 x_2)^2 - (u_1 x_1 + u_2 x_2) \\
&\rightarrow \min, \quad \text{s.t. } x_1 + x_2 = 1.
\end{align*}
\]
(12)

Let us denote
\[
L(x, \lambda) = (v_1 x_1 + v_2 x_2)^2 + (w_1 x_1 + w_2 x_2)^2
\]
and
\[
- (u_1 x_1 + u_2 x_2) + \lambda(x_1 + x_2 - 1),
\]
(13)

the Lagrange function of the constrained optimization problem (6). The Kuhn–Tucker necessity conditions are
\[
2v_1(v_1 x_1 + v_2 x_2) + 2w_1(w_1 x_1 + w_2 x_2) - u_1 + \lambda = 0,
\]
\[
2v_2(v_1 x_1 + v_2 x_2) + 2w_2(w_1 x_1 + w_2 x_2) - u_2 + \lambda = 0,
\]
x_1 + x_2 = 1,

which leads us to the following linear equation:
\[
[((v_1 - v_2)^2 + (w_1 - w_2)^2)x_1
\]
\[
= \frac{1}{2}(u_1 - u_2) - (v_1 - v_2)w_2 - (w_1 - w_2)v_2.
\]
(14)

If \((v_1 - v_2)^2 + (w_1 - w_2)^2 \neq 0\) then we find that \(x^* = (x_1^*, 1 - x_1^*)\), where
\[
x_1^* = \frac{1}{(v_1 - v_2)^2 + (w_1 - w_2)^2} \times \left[ \frac{1}{2}(u_1 - u_2) - (v_1 - v_2)w_2 - (w_1 - w_2)v_2 \right],
\]
(15)
is the unique solution to Eq. (14). If \( v_1 = v_2 \) and \( w_1 = w_2 \) then from (14) we find \( u_1 = u_2 \), which would contradict the initial assumption that the two assets are not identical. It can be easily seen that \( L''(x^*, \lambda) \) is a positive definite matrix in the subset defined by

\[
\{ y = (y_1, y_2) \in \mathbb{R}^2: y_1 + y_2 = 0 \}.
\]

So, \( x^* \) is the unique optimal solution to (12), and if \( x^* > 0 \) then \( x^* \) is an optimal solution to (4) with \( n = 2 \).

3. An algorithm

In this Section, we provide an algorithm for finding an optimal solution to the \( n \)-asset possibilistic portfolio selection problem (4). The algorithm will terminate in \( o(n^3) \) steps. Step 1: Let \( c := \infty \) and \( x_c := [0, \ldots, 0] \).

Step 2: Choose three points from the bag \( \{(v_i, w_i, u_i): i = 1, \ldots, n\} \) which have not been considered yet. If there are no such points then go to Step 9, otherwise denote these three points by \( (v_j, w_j, u_j), (v_k, w_k, u_k) \) and \( (v_l, w_l, u_l) \). Let \( (v_1, w_1, u_1) := (v_j, w_j, u_j), (v_2, w_2, u_2) := (v_k, w_k, u_k) \) and \( (v_3, w_3, u_3) := (v_l, w_l, u_l) \).

Step 3: If

\[
\det \begin{bmatrix} q_1 & r_1 \\ q_2 & r_2 \end{bmatrix} = \det \begin{bmatrix} v_1 - v_3 & w_1 - w_3 \\ v_2 - v_3 & w_2 - w_3 \end{bmatrix} = 0
\]

then go to Step 2, otherwise go to Step 4.

Step 4: Compute the first two component, \( [x^*_1, x^*_2] \), of the optimal solution to (6) using Eq. (10).

Step 5: If \( [x^*_1, x^*_2, 1 - x^*_2 - x^*_1] > 0 \) then go to Step 6, otherwise go to Step 2.

Step 6: If \( U_* < c \) then go to Step 7, otherwise go to Step 2.

Step 7: Let \( c = U_* \), where \( U_* \) is the optimal value of (6), and let

\[
x_c = [0, \ldots, x^*_1, 0, \ldots, 0, x^*_2, 0, \ldots, 0, x^*_3, 0, \ldots, 0].
\]

Step 8: Go to Step 2.

Step 9: Choose two points from the bag \( \{(v_i, w_i, u_i): i = 1, \ldots, n\} \) which have not been considered yet. If there are no such points then go to Step 16, otherwise denote these two points by \( (v_j, w_j, u_j) \) and \( (v_k, w_k, u_k) \). Let \( (v_1, w_1, u_1) := (v_j, w_j, u_j) \) and \( (v_2, w_2, u_2) := (v_k, w_k, u_k) \).

Step 10: If \( (v_1 - v_2)^2 + (w_1 - w_2)^2 \neq 0 \) then go to Step 9, otherwise go to Step 11.

Step 11: Compute the first component, \( x^*_1 \), of the optimal solution to (12) using Eq. (15).

Step 12: If \( [x^*_1, x^*_2] = [x^*_1, 1 - x^*_1] > 0 \) then go to Step 13, otherwise go to Step 9.

Step 13: If \( U_* < c \) then go to Step 14, otherwise go to Step 9.

Step 14: Let \( c = U_* \), where \( U_* \) is the optimal value of (12), and let

\[
x_c = [0, \ldots, x^*_1, 0, \ldots, 0, x^*_2, 0, \ldots, 0, x^*_3, 0, \ldots, 0].
\]

Step 15: Go to Step 9.

Step 16: Choose a point from the bag \( \{(v_i, w_i, u_i): i = 1, \ldots, n\} \) which has not been considered yet. If there is no such points then go to Step 20, otherwise denote this point by \( (v_2, w_2, u_2) \).

Step 17: If \( v^2_1 + w^2_1 - u_1 < c \) then go to Step 18, otherwise go to Step 16.

Step 18: Let \( c = v^2_1 + w^2_1 - u_1 \) and let

\[
x_c = [0, \ldots, 0, 1, 0, \ldots, 0].
\]

Step 19: Go to Step 16.

Step 20: \( x_c \) is an optimal solution and \( -c \) is the optimal value of the original portfolio selection problem (4).

4. Example

We shall illustrate the proposed algorithm by a simple example. Consider a 3-asset problem with \( A = 2.46 \) and with the following possibility distributions:

\[
r_1 = (-10.5, 70.0, 4.0, 100.0),
\]

\[
r_2 = (-8.1, 35.0, 4.4, 54.0),
\]

\[
r_3 = (-5.0, 28.0, 11.0, 85.0)
\]

and, therefore,

\[
(v_1, w_1, u_1) = (6.386, 1.359, 45.750),
\]

\[
(v_2, w_2, u_2) = (3.469, 0.763, 21.717),
\]

\[
(v_3, w_3, u_3) = (3.604, 1.255, 23.833).
\]
It should be noted that, the first asset may yield negative rates of return with degree of possibility one. Usually, the support of fuzzy numbers representing the possibility distributions of rates of return cannot contain any return that is \(-100\%\), because one can never lose more money than the original investment.

First consider the 3-asset problem with \((v_1, w_1, u_1), (v_2, w_2, u_2)\) and \((v_3, w_3, u_3)\). Since

\[
\det \begin{bmatrix} g_1 & r_1 \\ g_2 & r_2 \end{bmatrix} = \det \begin{bmatrix} 2.782 & 0.105 \\ -0.135 & -0.491 \end{bmatrix} = -1.352 \neq 0
\]

we get

\[
\begin{bmatrix} x^*_1 \\ x^*_2 \end{bmatrix} = \frac{1}{-1.352^2} \begin{bmatrix} 0.259 & 0.427 \\ 0.427 & 7.751 \end{bmatrix} \begin{bmatrix} 0.800 \\ 0.044 \end{bmatrix} = \begin{bmatrix} 0.124 \\ 0.373 \end{bmatrix}
\]

and, since,

\[
[x^*_1, x^*_2, x^*_3] = [0.124, 0.373, 0.503] > 0
\]

we get (Step 7)

\[
U_* := -9.386 \text{ and } x^* := [0.124, 0.373, 0.503].
\]

Thus, \([0.124, 0.373, 0.503]\) is a qualified candidate for an optimal solution to \((3)\).

Let us consider all conceivable 2-asset problems (1,2), (1,3) and (2,3), where the numbers stand for the corresponding assets chosen from the bag

\[
\{(v_1, w_1, u_1), (v_2, w_2, u_2), (v_3, w_3, u_3)\}\.
\]

Here we are searching for optimal solutions on the edges of the triangle generated by the assets.

**Select (1,2):** Since

\[
(v_1 - v_2)^2 + (w_1 - w_2)^2 = 8.864 \neq 0,
\]

we get

\[
U_* := -9.336 \text{ and } [x^*_1, x^*_2] = [0.163, 0.837] > 0.
\]

Thus, \([0.163, 0.837, 0]\) is a qualified candidate for an optimal solution to \((3)\).

**Select (1,3):** Since

\[
(v_1 - v_3)^2 + (w_1 - w_3)^2 = 7.751 \neq 0,
\]

we get

\[
U_* := -9.352 \text{ and } [x^*_1, x^*_3] = [0.103, 0.897] > 0.
\]

Thus, \([0.103, 0, 0.897]\) is a qualified candidate for an optimal solution to \((3)\).

**Select (2,3):** Since

\[
(v_2 - v_3)^2 + (w_2 - w_3)^2 = 0.259 \neq 0,
\]

we get

\[
U_* := -9.277 \text{ and } [x^*_2, x^*_3] = [0.171, 0.829] > 0.
\]

Thus, \([0, 0.171, 0.829]\) is a qualified candidate for an optimal solution to \((3)\).

Finally, we compute the utility values of all the three vertexes of the triangle generated by the three assets:

\[
v^*_1 + w^*_1 - u_1 = -3.122
\]

and \([1, 0, 0]\) is the corresponding feasible solution to \((3)\).

\[
v^*_2 + w^*_2 - u_2 = -9.101
\]

and \([0, 1, 0]\) is the corresponding feasible solution to \((3)\).

\[
v^*_3 + w^*_3 - u_3 = -9.269
\]

and \([0, 0, 1]\) is the corresponding feasible solution to \((3)\).

Comparing the utility values of all feasible solutions we find that the only solution to the 3-asset problem is \(x^* = [0.124, 0.373, 0.503]\) with a utility value of 9.386. The optimal risky portfolio will be preferred to the risk-free investment (by an investor whose degree of risk-aversion is equal to 2.46) if \(r_t < 9.386\%\).

**5. Summary**

In this paper, we have considered portfolio selection problems under possibility distributions and have presented an algorithm for finding an exact (i.e. not approximate) optimal solution to these problems. First, we have proved that the boundary of the set of feasible solutions (which is a convex polytope) must contain
all optimal solutions to the problem. Then we have considered all possible sides, edges and vertexes that could be generated from the given triplets and computed the optimal portfolios of: (i) three assets that could generate a side, and (ii) two assets that could generate an edge of the convex hull of all assets. Then we have compared the utility values of all feasible solutions (i.e. solutions with non-negative weights) and portfolios with the highest utility values have been chosen as optimal solutions to the portfolio selection problem.

References


