Capital budgeting techniques using discounted fuzzy versus probabilistic cash flows

Cengiz Kahraman \textsuperscript{a,*}, Da Ruan \textsuperscript{b}, Ethem Tolga \textsuperscript{c}

\textsuperscript{a} Department of Industrial Engineering, Istanbul Technical University, Maçka 80680, Istanbul, Turkey
\textsuperscript{b} Belgian Nuclear Research Center, Boeretang 200, B-2400 Mol, Belgium
\textsuperscript{c} Faculty of Engineering and Technology, Galatasaray University, 80840 Ortakoy, Istanbul, Turkey

Abstract

Risk analysis involves the development of the probability distribution for the measure of effectiveness. The risk associated with an investment alternative is generally either given as the possibility of an unfavorable value of the measure of effectiveness or measured by the variance of the measure of effectiveness. In an uncertain economic decision environment, an expert’s knowledge about discounting cash flows consists of a lot of vagueness instead of randomness. Cash amounts and interest rates are usually estimated by using educated guesses based on expected values or other statistical techniques to obtain them. Fuzzy numbers can capture the difficulties in estimating these parameters. In this paper, the formulas for the analyses of fuzzy present value, fuzzy equivalent uniform annual value, fuzzy future value, fuzzy benefit–cost ratio, and fuzzy payback period are developed and given some numeric examples. Then the examined cash flows are expanded to geometric and trigonometric cash flows and using these cash flows fuzzy present value, fuzzy future value, and fuzzy annual value formulas are developed for both discrete compounding and continuous compounding. © 2002 Elsevier Science Inc. All rights reserved.

Keywords: Capital budgeting; Fuzziness; Cash flow; Possibility; Probability
1. Introduction

When evaluating a risky project using decision and risk analysis techniques, analysts typically construct a model that calculates project cash flows and the net present value of these cash flows given different settings of input variables. To examine the risks of the projects, probability distributions are assigned to the input variables and a probability distribution is calculated, called a risk profile, showing the likelihood of the different possible net present values.

By invoking the central limit theorem, which credits a normal distribution to the sum of independently distributed random variables as the number of terms in the summation increases, it can be considered that present worth (PW) normally distributed with a mean of $E[PW]$ and a variance of $V(PW)$. Thus, a probability of having a loss or a gain from the investment can be easily calculated.

There are some other methods to consider risky cash flows such as certainty equivalent method and risk-adjusted discount rate method. These methods do not use a probability distribution.

In the following, first the main equations for probabilistic cash flows will be given and then capital budgeting techniques using fuzzy cash flows are explained and developed.

1.1. Probabilistic cash flows: the expected value and the variance of a probabilistic cash flow

Since the expected value of a sum of random variables equals the sum of the expected values of the random variables, then the expected present value (PV) is given by

$$E(PV) = \sum_{j=0}^{N} (1 + i)^{-j}E[A_j],$$

where $A_j$'s are statistically independent net cash flows and $N$ is the life of project. Then the variance of PV is given by

$$\sigma^2[PV] = V(PV) = \sum_{j=0}^{N} (1 + i)^{-2j}V(A_j).$$

The central limit theorem establishes that the sum of independently distributed random variables tends to be normally distributed as the number of terms in the summation increases. Hence, as $N$ increases, PV tends to be normally distributed with a mean value of $E[PV]$ and a variance of $V(PV)$.

In the case of a set of correlated cash flows ($A_j$'s are not statistically independent) the variance calculation is modified as follows:
\[ V[	ext{PV}] = \sum_{j=0}^{N} V(A_j)(1 + i)^{-2j} + 2 \sum_{j=0}^{N-1} \sum_{k=j+1}^{N} \text{Cov}[A_j, A_k](1 + i)^{-(j+k)}, \] (3)

where \( \text{Cov}[A_j, A_k] \) is the covariance between \( A_j \) and \( A_k \). \( \text{Cov}[A_j, A_k] \) equals \( \rho_{jk} \sigma[A_j] \sigma[A_k] \), where \( \rho_{jk} \) is the correlation coefficient between \( A_j \) and \( A_k \). If all \( A_j \) and \( A_k \) are perfectly correlated such that \( \rho_{jk} = +1 \), then

\[ V[	ext{PV}] = \left\{ \sum_{j=0}^{N} \sigma[A_j](1 + i)^{-j} \right\}^2. \] (4)

In performing risk analyses involving correlated cash flows, it is suggested that the net cash flow in a year be separated into those components of cash flow one can reasonably expect to be independent from year to year and those that are correlated over time.

1.2. Possibilistic cash flows: fuzzy sets and fuzzy numbers

To deal with vagueness of human thought, Zadeh [1] first introduced the fuzzy set theory, which was based on the rationality of uncertainty due to imprecision or vagueness. A major contribution of fuzzy set theory is its capability of representing vague knowledge. The theory also allows mathematical operators and programming to apply to the fuzzy domain.

A fuzzy number is a normal and convex fuzzy set with membership function \( \mu_i(x) \) which both satisfies normality: \( \mu_i(x) = 1 \), for at least one \( x \in R \) and convexity: \( \mu_i(x') \geq \mu_i(x_1) \land \mu_i(x_2) \), where \( \mu_i(x) \in [0, 1] \) and \( \forall x' \in [x_1, x_2] \). ‘\( \land \)’ stands for the minimization operator.

Quite often in finance future cash amounts and interest rates are estimated. One usually employs educated guesses, based on expected values or other statistical techniques, to obtain future cash flows and interest rates. Statements like \( \text{approximately between } \$12,000 \text{ and } \$16,000 \) or \( \text{approximately between } 10\% \text{ and } 15\% \) must be translated into an exact amount, such as \$14,000 or 12.5%, respectively. Appropriate fuzzy numbers can be used to capture the vagueness of those statements.

A tilde will be placed above a symbol if the symbol represents a fuzzy set. Therefore, \( \tilde{P}, \tilde{F}, \tilde{G}, \tilde{A}, \tilde{i}, \tilde{r} \) are all fuzzy sets. The membership functions for these fuzzy sets will be denoted by \( \mu(x|\tilde{P}), \mu(x|\tilde{F}), \mu(x|\tilde{G}) \), etc. A fuzzy number is a special fuzzy subset of the real numbers. The extended operations of fuzzy numbers are given in Appendix A. A triangular fuzzy number (TFN) is shown in Fig. 1. The membership function of a TFN (\( \tilde{M} \)) defined by

\[ \mu(x|\tilde{M}) = (m_1, f_1(y|\tilde{M})/m_2, m_2/f_2(y|\tilde{M}), m_3), \] (5)

where \( m_1 < m_2 < m_3 \), \( f_1(y|\tilde{M}) \) is a continuous monotone increasing function of \( y \) for \( 0 \leq y \leq 1 \) with \( f_1(0|\tilde{M}) = m_1 \) and \( f_1(1|\tilde{M}) = m_2 \) and \( f_2(y|\tilde{M}) \) is a continuous
monotone decreasing function of $y$ for $0 \leq y \leq 1$ with $f_2(0|\bar{M}) = m_3$ and $f_2(1|\bar{M}) = m_2$. $\mu(x|\bar{M})$ is denoted simply as $(m_1/m_2, m_2/m_3)$.

A flat fuzzy number (FFN) is shown in Fig. 2. The membership function of an FFN, $\bar{V}$, is defined by

$$\mu(x|\bar{V}) = (m_1, f_1(y|\bar{V})/m_2, m_3/f_2(y|\bar{V}), m_4),$$

(6)

where $m_1 < m_2 < m_3 < m_4$, $f_1(y|\bar{V})$ is a continuous monotone increasing function of $y$ for $0 \leq y \leq 1$ with $f_1(0|\bar{V}) = m_1$ and $f_1(1|\bar{V}) = m_2$ and $f_2(y|\bar{V})$ is a continuous monotone decreasing function of $y$ for $0 \leq y \leq 1$ with $f_2(1|\bar{V}) = m_3$ and $f_2(0|\bar{V}) = m_4$. $\mu(y|\bar{V})$ is denoted simply as $(m_1/m_2, m_3/m_4)$.

The fuzzy sets $\bar{P}, \bar{F}, \bar{G}, \bar{A}, \bar{i}, \bar{r}$ are usually fuzzy numbers but $n$ will be discrete positive fuzzy subset of the real numbers [2]. The membership function $\mu(x|\bar{n})$ is defined by a collection of positive integers $n_i$, $1 \leq i \leq K$, where

$$\mu(x|\bar{n}) = \left\{ \begin{array}{ll} \mu(n_i|\bar{n}) = \lambda_i, & 0 \leq \lambda_i \leq 1, \\ 0 & \text{otherwise}. \end{array} \right.$$  

(7)
2. Fuzzy present-value method

The present-value method of alternative evaluation is very popular because future expenditures or receipts are transformed into equivalent dollars now. That is, all of the future cash flows associated with an alternative are converted into present dollars. If the alternatives have different lives, the alternatives must be compared over the same number of years.

Chiu and Park [3] propose a present-value formulation of a fuzzy cash flow. The result of the present value is also a fuzzy number with nonlinear membership function. The present value can be approximated by a TFN. Chiu and Park’s [3] formulation is

\[
\tilde{P}V = \left( \sum_{t=0}^{n} \left( \frac{\max(P^{l(t)}_t, 0)}{\prod_{t'=0}^{t}(1 + r^{l(t')}_t)} + \frac{\min(P^{r(t)}_t, 0)}{\prod_{t'=0}^{t}(1 + r^{r(t')}_t)} \right) \right),
\]

\[
\tilde{P}V = \left( \sum_{t=0}^{n} \left( \frac{\max(P^{l(t)}_t, 0)}{\prod_{t'=0}^{t}(1 + r^{l(t')}_t)} + \frac{\min(P^{r(t)}_t, 0)}{\prod_{t'=0}^{t}(1 + r^{r(t')}_t)} \right) \right),
\]

where \( P^{l(t)}_t \) is the left representation of the cash at time \( t \), \( P^{r(t)}_t \) is the right representation of the cash at time \( t \), \( r^{l(t)}_t \) is the left representation of the interest rate at time \( t \), and \( r^{r(t)}_t \) is the right representation of the interest rate at time \( t \).

Buckley’s [2] membership function for \( \tilde{P}_n \),

\[
\mu(x|\tilde{P}_n) = (p_{n_1}, f_{n_1}(y|\tilde{P}_n), p_{n_2}, p_{n_3})
\]

is determined by

\[
f_{n_i}(y|\tilde{P}_n) = f_i(y|\tilde{F})(1 + f_k(y|\tilde{F}))^{-n}
\]

for \( i = 1, 2 \), where \( k = i \) for negative \( \tilde{F} \) and \( k = 3 - i \) for positive \( \tilde{F} \).

Ward [4] gives the fuzzy present-value function as

\[
\tilde{P}V = (1 + r)^{-n}(a, b, c, d),
\]

where \( (a, b, c, d) \) is a flat fuzzy filter function (4F) number.

**Example 1.** A $(-14,000, -12,000, -10,000) investment will return annual benefits of $(2,650, 2,775, 2,900) for six years with no salvage value at the end of six years. Compute the fuzzy present worth of the cash flow using an interest of (7.12%, 10.25%, 13.42%) per year.

\[
f_{6,1}(y|\tilde{P}) = \sum_{j=0}^{6} f_{j,1}(y|\tilde{F})(1 + f_{k(j)}(y|\tilde{F}))^{-j},
\]

\[
f_{6,2}(y|\tilde{P}) = \sum_{j=0}^{6} f_{j,2}(y|\tilde{F})(1 + f_{k(j)}(y|\tilde{F}))^{-j},
\]
for \( i = 1, 2 \) where \( k(j) = i \) for negative \( \tilde{F}_j \) and \( k(j) = 3 - i \) for positive \( \tilde{F}_j \).

For \( y = 0 \), \( f_{6.1}(y | \tilde{P}) = $ - 3525.57 \).
For \( y = 1 \), \( f_{6.1}(y | \tilde{P}) = f_{6.2}(y | \tilde{P}) = $ - 24.47 \).
For \( y = 0 \), \( f_{6.2}(y | \tilde{P}) = $ + 3786.34 \).

The possibility of \( NPV = 0 \) for this triangular fuzzy number can be calculated using a linear interpolation:

\[
x = -3810.81y + 3786.34
\]

For \( x = 0 \), \( \text{Poss}(NPV = 0) = 0.9936 \).

3. Fuzzy capitalized value method

A specialized type of cash flow series is a perpetuity, a uniform series of cash flows which continues indefinitely. An infinite cash flow series may be appropriate for such very long-term investment projects as bridges, highways, forest harvesting, or the establishment of endowment funds where the estimated life is 50 years or more.

In the nonfuzzy case, if a present value \( P \) is deposited into a fund at interest rate \( r \) per period so that a payment of size \( A \) may be withdrawn each and every period forever, then the following relation holds between \( P, A, \) and \( r \):

\[
P = \frac{A}{r}.
\]  

(14)

In the fuzzy case, let us assume all the parameters as triangular fuzzy numbers: \( \tilde{P} = (p_1, p_2, p_3) \) or \( \tilde{P} = ((p_2 - p_1)y + p_1, (p_2 - p_3)y + p_3) \) and \( \tilde{A} = (a_1, a_2, a_3) \) or \( \tilde{A} = ((a_2 - a_1)y + a_1, (a_2 - a_3)y + a_3) \) and \( \tilde{r} = (r_1, r_2, r_3) \) or \( \tilde{r} = ((r_2 - r_1)y + r_1, (r_2 - r_3)y + r_3) \), where \( y \) is the membership degree of a certain point of \( A \) and \( r \) axis. If \( \tilde{A} \) and \( \tilde{r} \) are both positive,

\[
\tilde{P} = \tilde{A} \ominus \tilde{r} = (a_1/r_3, a_2/r_2, a_3/r_1)
\]  

(15)

or

\[
\tilde{P} = (((a_2 - a_1)y + a_1)/(r_2 - r_3)y + r_3), ((a_2 - a_3)y + a_3)/(r_2 - r_1)y + r_1)).
\]  

(16)

If \( \tilde{A} \) is negative and \( \tilde{r} \) is positive,

\[
\tilde{P} = \tilde{A} \ominus \tilde{r} = (a_1/r_1, a_2/r_2, a_3/r_3)
\]  

(17)

or

\[
\tilde{P} = (((a_2 - a_1)y + a_1)/(r_2 - r_1)y + r_1), ((a_2 - a_3)y + a_3)/(r_2 - r_3)y + r_3)).
\]  

(18)
Now, let $\tilde{A}$ be an expense every $n$th period forever, with the first expense occurring at $n$. For example, an expense of $(\$5000, \$7000, \$9000)$ every third year forever, with the first expense occurring at $t = 3$. In this case, the fuzzy effective rate $\bar{e}$ may be used as in the following:

$$ f_1(\bar{y}|\bar{e}) = (1 + (1/m)f_1(\bar{y}|\bar{r}'))^m - 1, $$

where $i = 1, 2, f_1(\bar{y}|\bar{e})$ is a continuous monotone increasing function of $\bar{y}; f_2(\bar{y}|\bar{e})$ is a continuous monotone decreasing function of $\bar{y}; m$ is the number of compoundings per period; and $\bar{r}'$ is the fuzzy nominal interest rate per period. The membership function of $\bar{e}$ may be given as

$$ \mu(x|\bar{e}) = (e_1, f_1(\bar{y}|\bar{e})/e_2, e_2/f_2(\bar{y}|\bar{e}), e_3). $$

If $\tilde{A}$ and $f_i(\bar{y}|\bar{e})$ are both positive,

$$ \tilde{P} = \tilde{A} \oplus \bar{e} = (((a_2 - a_1)y + a_1)/f_2(\bar{y}|\bar{e}), ((a_2 - a_3)y + a_3)/f_1(\bar{y}|\bar{e})). \quad (21) $$

If $\tilde{A}$ is negative and $f_i(\bar{y}|\bar{e})$ is positive,

$$ \tilde{P} = \tilde{A} \ominus \bar{e} = (((a_2 - a_1)y + a_1)/f_1(\bar{y}|\bar{e}), ((a_2 - a_3)y + a_3)/f_2(\bar{y}|\bar{e})), \quad (22) $$

$(a_2 - a_1)y + a_1$ and $(a_2 - a_3)y + a_3$ can be symbolized as $f_1(\bar{y}|\bar{a})$ and $f_2(\bar{y}|\bar{a})$, respectively.

**Example 2.** Project ABC consists of the following requirements. Find the capitalized worth of the project if $\bar{r} = (12\%, 15\%, 18\%)$ annually.

1. A $(\$40,000, \$50,000, \$60,000)$ first cost ($\tilde{FC}$) at $t = 0$,
2. A $(\$4000, \$5000, \$6000)$ expense ($\tilde{A}_1$) every day,
3. A $(\$20,000, \$25,000, \$30,000)$ expense ($\tilde{A}_2$) every third year forever, with the first expense occurring at $t = 3$. See Fig. 3.

$$ \tilde{P} = \tilde{FC} \oplus (\tilde{A}_1 \ominus \bar{r}) + (\tilde{A}_2 \ominus \bar{e}), \quad (23) $$

$$ \tilde{P} = ((10,000y + 40,000), (-10,000y + 60,000)) $$

$$ + ((1000y + 4000)/(-0.03y + 0.18), $$

$$ (-1000y + 6000)/(0.03y + 0.12)) $$

$$ + ((5000y + 20,000)/(1.18 - 0.03y)^3 - 1), $$

$$ (-5000y + 30,000)/(1.12 + 0.03y)^3 - 1). $$

![Fig. 3. The cash flow diagram for Example 2.](image-url)
4. Fuzzy future value method

The future value (FV) of an investment alternative can be determined by using the relationship

$$FV(r) = \sum_{t=0}^{n} P_t (1 + i)^{n-t},$$

(24)

where $FV(r)$ is defined as the future value of the investment using a minimum attractive rate of return (MARR) of $r\%$. The future-value method is equivalent to the present-value method and the annual-value method.

Chiu and Park’s [3] formulation for the fuzzy future value has the same logic of fuzzy present-value formulation:

$$\bar{FV} = \left\{ \sum_{t=0}^{n-1} \left[ \max \left( P_t^{(y)}, 0 \right) \prod_{t'=t+1}^{n} \left( 1 + r_{t'}^{(y)} \right) \right] $$

$$+ \min \left( P_t^{(y)}, 0 \right) \prod_{t'=t+1}^{n} \left( 1 + r_{t'}^{(y)} \right) \right] + P_{n}^{(y)}, $$

$$\sum_{t=0}^{n-1} \left[ \max \left( P_t^{(y)}, 0 \right) \prod_{t'=t+1}^{n} \left( 1 + r_{t'}^{(y)} \right) $$

$$+ \min \left( P_t^{(y)}, 0 \right) \prod_{t'=t+1}^{n} \left( 1 + r_{t'}^{(y)} \right) \right] + P_{n}^{(y)} \right\}. $$

(25)

Buckley’s [2] membership function $\mu(x|\tilde{F})$ is determined by

$$f_i(y|\tilde{F}_n) = f_i(y|\tilde{P})(1 + f_i(y|\tilde{r}))^n. $$

(26)

For the uniform cash flow series, $\mu(x|\tilde{F})$ is determined by

$$f_n(y|\tilde{F}) = f_i(y|\tilde{A}) \beta(n, f_i(y|\tilde{r})), $$

(27)

where $i = 1, 2$ and $\beta(n, r) = (((1 + r)^n - 1)/r)$ and $\tilde{A} > 0$ and $\tilde{r} > 0$.

5. Fuzzy benefit/cost ratio method

The benefit/cost ratio (BCR) is often used to assess the value of a municipal project in relation to its cost; it is defined as

For $y = 0$, $f_{x_1}(y|\tilde{P}) = $93326.42.
For $y = 1$, $f_{x_2}(y|\tilde{P}) = f_{x_2}(y|\tilde{P}) = $131317.97.
For $y = 0$, $f_{x_2}(y|\tilde{P}) = $184074.07.
BCR = \frac{B - D}{C}, \tag{28}

where \(B\) represents the equivalent value of the benefits associated with the project, \(D\) represents the equivalent value of the disbenefits, and \(C\) represents the project’s net cost. A BCR greater than 1.0 indicates that the project evaluated is economically advantageous. In BCR analyses, costs are not preceded by a minus sign.

When only one alternative must be selected from two or more mutually exclusive (stand-alone) alternatives, a multiple alternative evaluation is required. In this case, it is necessary to conduct an analysis on the incremental benefits and costs. Suppose that there are two mutually exclusive alternatives. In this case, for the incremental BCR analysis ignoring disbenefits the following ratios must be used:

\[ \frac{\Delta B_{2-1}}{\Delta C_{2-1}} = \frac{\Delta \text{PVB}_{2-1}}{\Delta \text{PVC}_{2-1}}, \tag{29} \]

where \(\Delta \text{PVB}\) is the present value of benefits, and \(\Delta \text{PVC}\) is the present value of costs. If \(\frac{\Delta B_{2-1}}{\Delta C_{2-1}} \leq 1.0\), Alternative 2 is preferred.

In the case of fuzziness, first, it will be assumed that the largest possible value of Alternative 1 for the cash in year \(t\) is less than the least possible value of Alternative 2 for the cash in year \(t\). The fuzzy incremental BCR is

\[ \frac{\Delta \tilde{B}}{\Delta \tilde{C}} = \left( \frac{\sum_{t=0}^{n}(B_{2t}^{\tilde{y}} - B_{1t}^{\tilde{y}})(1 + r_{1}^{\tilde{y}})^{-t}}{\sum_{t=0}^{n}(C_{2t}^{\tilde{y}} - C_{1t}^{\tilde{y}})(1 + r_{1}^{\tilde{y}})^{-t}}, \right) \cdot \frac{\sum_{t=0}^{n}(\Delta B_{2t}^{\tilde{y}} - \Delta B_{1t}^{\tilde{y}})(1 + r_{1}^{\tilde{y}})^{-t}}{\sum_{t=0}^{n}(\Delta C_{2t}^{\tilde{y}} - \Delta C_{1t}^{\tilde{y}})(1 + r_{1}^{\tilde{y}})^{-t}}. \tag{30} \]

If \(\frac{\Delta \tilde{B}}{\Delta \tilde{C}}\) is equal to or greater than \((1, 1, 1)\), Alternative 2 is preferred.

In the case of a regular annuity, the fuzzy \(\frac{\tilde{B}}{\tilde{C}}\) ratio of a single investment alternative is

\[ \frac{\tilde{B}}{\tilde{C}} = \left( \frac{\tilde{A}^{\tilde{y}}(n, r^{\tilde{y}})}{\tilde{C}^{\tilde{y}}}, \frac{\tilde{A}^{\tilde{y}}(n, r^{\tilde{y}})}{\tilde{C}^{\tilde{y}}}, \right) \tag{31} \]

where \(\tilde{C}\) is the first cost and \(\tilde{A}\) is the net annual benefit, and \(\gamma(n, r) = ((1 + r)^{n} - 1)/(1 + r)^{n}r\). The \(\frac{\Delta \tilde{B}}{\Delta \tilde{C}}\) ratio in the case of a regular annuity is [6]

\[ \frac{\Delta \tilde{B}}{\Delta \tilde{C}} = \left( \frac{(A_{2}^{\tilde{y}} - A_{1}^{\tilde{y}})\gamma(n, r^{\tilde{y}})}{C_{2}^{\tilde{y}} - C_{1}^{\til{y}}}, \frac{(A_{2}^{\til{y}} - A_{1}^{\til{y}})\gamma(n, r^{\til{y}})}{C_{2}^{\til{y}} - C_{1}^{\til{y}}} \right). \tag{32} \]
6. Fuzzy equivalent uniform annual value method

The equivalent uniform annual value (EUAV) means that all incomes and disbursements (irregular and uniform) must be converted into an equivalent uniform annual amount, which is the same each period. The major advantage of this method overall the other methods is that it does not require making the comparison over the least common multiple of years when the alternatives have different lives [5]. The general equation for this method is

\[ \text{EUAV} = A = \text{NPV} \gamma^{-1}(n, r) = \frac{\text{NPV}(1 + r)^n - 1}{(1 + r)^n} \]

where NPV is the net present value. In the case of fuzziness, \( \tilde{\text{NPV}} \) will be calculated and then the fuzzy EUAV (\( \tilde{A} \)) will be found. The membership function \( \mu(x|\tilde{A}) \) for \( \tilde{A} \) is determined by

\[ f_n(y|\tilde{A}) = f(y|\tilde{\text{NPV}}) \gamma^{-1}(n, f(y|\tilde{r})) \]

and TFN\( (y) \) for fuzzy EUAV is

\[ \tilde{A}(y) = \left( \frac{\text{NPV}_{\text{f}(y)}}{\gamma(n, \tilde{r}(y))}, \frac{\text{NPV}_{\text{r}(y)}}{\gamma(n, \tilde{r}(y))} \right) \]

Example 3. Assume that \( \tilde{\text{NPV}} = (-83525.57, $24.47, +$3786.34) \) and \( \tilde{r} = (3\%, 5\%, 7\%) \). Calculate the fuzzy EUAV:

\[ f_{6,1}(y|\tilde{A}_6) = (3501.1y - 3525.57) \frac{(1.03 + 0.02y)^6(0.02y + 0.03)}{(1.03 + 0.02y)^6 - 1}, \]

\[ f_{6,2}(y|\tilde{A}_6) = (-3810.81y + 3786.34) \frac{(1.07 - 0.02y)^6(0.07 - 0.02y)}{(1.07 - 0.02y)^6 - 1}. \]

For \( y = 0 \), \( f_{6,1}(y|\tilde{A}_6) = -$650.96. For \( y = 1 \), \( f_{6,1}(y|\tilde{A}_6) = f_{6,2}(y|\tilde{A}_6) = -$4.82. For \( y = 0 \), \( f_{6,2}(y|\tilde{A}_6) = +$795.13. \)

7. Fuzzy payback period method

The payback period method involves the determination of the length of time required to recover the initial cost of investment based on a zero interest rate ignoring the time value of money or a certain interest rate recognizing the time value of money. Let \( C_{j0} \) denote the initial cost of investment alternative \( j \), and \( R_{jt} \) denote the net revenue received from investment \( j \) during period \( t \). Assuming no other negative net cash flows occur, the smallest value of \( m_j \) ignoring the time value of money such that
\[
\sum_{j=1}^{m_j} R_{j\ell} \geq C_{j0}
\]  
(36)

or the smallest value of \( m_j \) recognizing the time value of money such that

\[
\sum_{j=1}^{m_j} R_{j\ell} (1 + r)^{-t} \geq C_{j0}
\]  
(37)

define the payback period for the investment \( j \). The investment alternative having the smallest payback period is the preferred alternative. In the case of fuzziness, the smallest value of \( m_j \) ignoring the time value of money such that

\[
\left( \sum_{j=1}^{m_j} r_{1,j\ell}, \sum_{j=1}^{m_j} r_{2,j\ell}, \sum_{j=1}^{m_j} r_{3,j\ell} \right) \geq (C_{1,0}, C_{2,0}, C_{3,0})
\]  
(38)

and the smallest value of \( m_j \) recognizing the time value of money such that

\[
\left( \sum_{j=1}^{m_j} (R_{j\ell}^{(y)}/(1 + r^{(y)}))^{t}, \sum_{j=1}^{m_j} (R_{j\ell}^{(y)}/(1 + r^{(y)})^{t} \right) \\
\geq ((C_{2,0} - C_{1,0}) y + C_{1,0}, (C_{2,0} - C_{3,0}) y + C_{3,0})
\]  
(39)

define the payback period for investment \( j \), where \( r_{k,j\ell} \) is the \( k \)th parameter of a triangular fuzzy \( R_{j\ell} \); \( C_{k,j0} \) is the \( k \)th parameter of a triangular fuzzy \( C_{j0} \); \( R_{j\ell}^{(y)} \) is the left representation of a triangular fuzzy \( R_{j\ell} \); \( R_{j\ell}^{(y)} \) is the right representation of a triangular fuzzy \( R_{j\ell} \). If it is assumed that the discount rate changes from one period to another, \((1 + r^{(y)})^{t} \) and \((1 + r^{(y)})^{t} \) will be \( \prod_{t=1}^{t} (1 + r_{t}^{(y)}) \) and \( \prod_{t=1}^{t} (1 + r_{t}^{(y)}) \), respectively.

It is now necessary to use a ranking method to rank the triangular fuzzy numbers such as Chiu and Park’s [3] method, Chang’s [7] method, Dubois and Prade’s [8] method, Jain’s [9] method, Kaufmann and Gupta’s [10] method, Yager’s [11] method. These methods may give different ranking results and most methods are tedious in graphic manipulation requiring complex mathematical calculation. In the following, two of the methods which do not require graphical representations are given. Chiu and Park’s [3] weighted method for ranking TFNs with parameters \((a, b, c)\) is formulated as

\[
((a + b + c)/3) + wb,
\]  
(40)

where \( w \) is a value determined by the nature and the magnitude of the most promising value. The preference of projects is determined by the magnitude of this sum.
Kaufmann and Gupta [10] suggest three criteria for ranking TFNs with parameters \((a, b, c)\). The dominance sequence is determined according to priority of:

1. Comparing the ordinary number \((a + 2b + c)/4\).
2. Comparing the mode (the corresponding most promise value), \(b\), of each TFN.
3. Comparing the range, \(c - a\), of each TFN.

The preference of projects is determined by the amount of their ordinary numbers. The project with the larger ordinary number is preferred. If the ordinary numbers are equal, the project with the larger corresponding most promising value is preferred. If projects have the same ordinary number and most promising value, the project with the larger range is preferred.

**Example 4.** Assume that there are two alternative machines that are under consideration to replace an aging production machine. The associated cash flows are given in the following table. Determine the best alternative by using the payback period method recognizing the time value of money. The fuzzy interest rate is \((12\%, 15\%, 18\%)\) annually. See Table 1.

If Chiu and Park’s [3] method (CP) is used for ranking TFNs, it is obtained \((w = 0.3)\):

For Alternative \(A\),

\[
CP_0 = \left( \frac{C_{1\mu} + C_{2\mu} + C_{3\mu}}{3} \right) + wC_{2\mu} = -6500, \\
TFN_1 = \left( \frac{1000y + 2000}{1.18 - 0.03y} \cdot \frac{-1000y + 4000}{1.12 + 0.03y} \right) = (1695, 2608.7, 3571.4), \\
CP_1 = 3407.6, \\
TFN_2 = \left( \frac{500y + 4000}{(1.18 - 0.03y)^2} \cdot \frac{-500y + 5000}{(1.12 + 0.03y)^2} \right) = (2872.7, 3402.6, 3986), \\
CP_2 = 4441.2, \\
\sum_{i=1}^{2} CP_i \succ CP_0 \rightarrow PP_A = 1.656 \text{ years}.
\]

For Alternative \(B\),

<table>
<thead>
<tr>
<th>End of year</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alternative (A)</td>
<td>((-7, -5, -3))</td>
<td>((2, 3, 4))</td>
<td>((4, 4.5, 5))</td>
<td>((1, 1.5, 2))</td>
<td>((3.5, 4, 4.5))</td>
</tr>
<tr>
<td>Alternative (B)</td>
<td>((-12, -10, -8))</td>
<td>((3, 4, 5))</td>
<td>((4.5, 5, 5.5))</td>
<td>((3.3, 5, 4))</td>
<td>((4, 4.5, 5))</td>
</tr>
</tbody>
</table>
\[ \begin{align*}
CP_0 &= -13,000, \\
TFN_1 &= (2542.4, 3478.3, 4464.3),
CP_1 &= 4538.5, \\
TFN_2 &= (3231.8, 3780.7, 4384.6), \\
CP_2 &= 4993.2, \\
TFN_3 &= (1825.9, 2301.3, 2847.1), \\
CP_3 &= 3015.2, \\
TFN_4 &= (2063.2, 2572.9, 3177.6), \\
CP_4 &= 3376.4,
\end{align*} \]

\[ \sum_{i=1}^{4} CP_i \succ CP_0 \rightarrow PP_B = 3.3 \text{ years}. \]

Alternative \( A \) is the preferred one.

8. Fuzzy internal rate of return method

The internal rate of return (IRR) method is referred to in the economic analysis literature as the discounted cash flow rate of return, internal rate of return, and the true rate of return. The internal rate of return on an investment is defined as the rate of interest earned on the unrecovered balance of an investment. Letting \( r^* \) denote the rate of return, the equation for obtaining \( r^* \) is

\[ \sum_{i=1}^{n} P_i (1 + r^*)^{-i} - FC = 0, \quad (41) \]

where \( P_i \) is the net cash flow at the end of period \( t \).

Assume the cash flow \( \tilde{F} = \tilde{F}_0, \tilde{F}_1, \ldots, \tilde{F}_n \) is fuzzy. \( \tilde{F}_n \) is a negative fuzzy number and the other \( \tilde{F}_i \) may be positive or negative fuzzy numbers. The fuzzy \( IRR(\tilde{F}, n) \) is a fuzzy interest rate \( \tilde{r} \) that makes the present value of all future cash amounts equal to the initial cash outlay. Therefore, the fuzzy number \( \tilde{r} \) satisfies

\[ \sum_{i=1}^{n} PV_{k(i)}(\tilde{F}_i, i) = -\tilde{F}_0, \quad (42) \]

where \( \sum \) is fuzzy addition, \( k(i) = 1 \) if \( \tilde{F}_i \) is negative and \( k(i) = 2 \) if \( \tilde{F}_i \) is positive.

Buckley [2] shows that such simple fuzzy cash flows may not have a fuzzy IRR and concludes that the IRR technique does not extend to fuzzy cash flows. Ward [4] considers Eq. (41) and explains that such a procedure cannot be
applied for the fuzzy case because the RHS of Eq. (41) is fuzzy, 0 is crisp, and an equality is impossible.

9. An expansion to geometric and trigonometric cash flows

When the value of a given cash flow differs from the value of the previous cash flow by a constant percentage, \( j \% \), then the series is referred to as a geometric series. If the value of a given cash flow differs from the value of the previous cash flow by a sinusoidal wave or a cosinusoidal wave, then the series is referred to as a trigonometric series.


The present value of a crisp geometric series is given by

\[
P = \sum_{n=1}^{N} F_1 (1 + g)^{n-1} (1 + i)^{-n} = \frac{F_1}{1 + g} \sum_{n=1}^{N} \left( \frac{1 + g}{1 + i} \right)^n , \tag{43}
\]

where \( F_1 \) is the first cash at the end of the first year. When this sum is made, the following present value equation is obtained:

\[
P = \begin{cases} 
F_1 \frac{1 - (1 + g)^N (1 + i)^{-N}}{i - g}, & i \neq g, \\
\frac{N F_1}{1 + i}, & i = g 
\end{cases} \tag{44}
\]

and the future value is

\[
F = \begin{cases} 
F_1 \frac{(1 + i)^N - (1 + g)^N}{i - g}, & i \neq g, \\
NF_1 (1 + i)^{N-1}, & i = g.
\end{cases} \tag{45}
\]

In the case of fuzziness, the parameters used in Eq. (44) will be assumed to be fuzzy numbers, except project life. Let

\[
\gamma(i, g, N) = \frac{1 - (1 + g)^N (1 + i)^{-N}}{i - g}, \quad i \neq g.
\]

As it is in Figs. 1 and 2, when \( k = 1 \), the left side representation will be depicted and when \( k = 2 \), the right side representation will be depicted. In this case, for \( i \neq g \)

\[
f_{n_k}(y|\tilde{P}_n) = f_k(y|\tilde{F}_1)\gamma(f_{n-k}(y|\tilde{i}), f_{n-k}(y|\tilde{g}), N). \tag{46}
\]

In Eq. (46), the least possible value is calculated for \( k = 1 \) and \( y = 0 \); the largest possible value is calculated for \( k = 2 \) and \( y = 0 \); the most promising value is calculated for \( k = 1 \) or \( k = 2 \) and \( y = 1 \).
To calculate the future value of a fuzzy geometric cash flow, let
\[
\zeta(i, g, N) = \frac{(1 + i)^N - (1 + g)^N}{i - g}, \quad i \neq g.
\]

Then the fuzzy future value is
\[
f_{Nk}(y|\bar{F}_N) = f_k(y|\bar{F}_1) \zeta(f_k(y|\bar{i}), f_k(y|\bar{g}), N). \tag{47}
\]

In Eq. (47), the least possible value is calculated for \(k = 1\) and \(y = 0\); the largest possible value is calculated for \(k = 2\) and \(y = 0\); the most promising value is calculated for \(k = 1\) or \(k = 2\) and \(y = 1\). This is also valid for the formulas developed at the rest of the paper.

The fuzzy uniform equivalent annual value can be calculated by using Eq. (48):
\[
f_{Nk}(y|\bar{A}) = f_k(y|\bar{P}_N) \vartheta(f_k(y|\bar{i}), N), \tag{48}
\]
where \(\vartheta(i, N) = ((1 + i)^N i)/((1 + i)^N - 1)\) and \(f(y|\bar{P}_N)\) is the fuzzy present value of the fuzzy geometric cash flows.

### 9.2. Geometric series – fuzzy cash flows in continuous compounding

In the case of crisp sets, the present and future values of discrete payments are given by Eqs. (49) and (50), respectively:
\[
P = \begin{cases} 
F_1 \frac{1 - e^{(r-g)N}}{e^r - e^r}, & r \neq g, \\
NF_1, & g = e^r - 1,
\end{cases} \tag{49}
\]
\[
F = \begin{cases} 
F_1 \frac{e^{rN} - e^gN}{e^r - e^r}, & r \neq g, \\
NF_1 e^{r(N-1)}, & g = e^r - 1
\end{cases} \tag{50}
\]
and the present and future values of continuous payments are given by Eqs. (51) and (52), respectively:
\[
P = \begin{cases} 
F_1 \frac{1 - e^{N(r-g)}}{r-g}, & r \neq g, \\
NF_1, & r = g,
\end{cases} \tag{51}
\]
\[
F = \begin{cases} 
F_1 \frac{e^{gN} - e^{gN}}{r-g}, & r \neq g, \\
NF_1 e^{gN}, & r = g
\end{cases} \tag{52}
\]

The fuzzy present and future values of the fuzzy geometric discrete cash flows in continuous compounding can be given as in Eqs. (53) and (54), respectively:
\[
f_{Nk}(y|\bar{P}_N) = f_k(y|\bar{F}_1) \beta(f_{3-k}(y|\bar{i}), f_{3-k}(y|\bar{g}), N), \tag{53}
\]
\[
f_{Nk}(y|\bar{F}) = f_k(y|\bar{F}_1) \tau(f_k(y|\bar{i}), f_k(y|\bar{g}), N), \tag{54}
\]
where
\[ \beta(r, g, N) = F_1 \left( (1 - e^{(g-r)N})/(e^r - e^g) \right), \quad r \neq g \]
for present value and
\[ \tau(r, g, N) = F_1 \left( (e^{rN} - e^g)/(e^r - e^g) \right), \quad r \neq g \]
for future value.

The fuzzy present and future values of the fuzzy geometric continuous cash flows in continuous compounding can be given as in Eqs. (55) and (56), respectively:
\[ f_{Nk}(y|\tilde{P}_N) = f_k(y|\tilde{F}_1) \eta(f_{3-k}(y|\tilde{r}), f_{3-k}(y|\tilde{g}), N), \quad (55) \]
\[ f_{Nk}(y|\tilde{F}_N) = f_k(y|\tilde{F}_1) v(f_k(y|\tilde{r}), f_k(y|\tilde{g}), N), \quad (56) \]
where
\[ \eta(r, g, N) = F_1 \frac{1 - e^{(g-r)N}}{r-g}, \quad v(r, g, N) = F_1 \frac{e^{rN} - e^g}{r-g}, \quad r \neq g. \]

9.3. Trigonometric series – fuzzy continuous cash flows

In Fig. 4, the function of the semi-sinusodial wave cash flows is depicted. This function, \( h(t) \), is given by Eq. (57) in the crisp case:
\[ h(t) = \begin{cases} D \sin \pi t, & 0 \leq t \leq 1, \\ 0 & \text{otherwise}. \end{cases} \quad (57) \]
The future value of a semi-sinusoidal cash flow for \( T = 1 \) and \( g \) is defined by Eq. (58):
\[ V(g, 1) = D \int_0^1 e^{(1-t)} \sin \pi t \, dt = D \frac{\pi (2 + g)}{r^2 + \pi^2}. \quad (58) \]

Fig. 5 shows the function of a cosinusoidal wave cash flow. This function, \( h(t) \), is given by Eq. (59):
\[ h(t) = \begin{cases} D (\cos 2\pi t + 1), & 0 \leq t \leq 1, \\ 0 & \text{otherwise}. \end{cases} \quad (59) \]

![Fig. 4. Semi-sinusoidal wave cash flow function.](image-url)
The future value of a cosinusoidal cash flow for $T = 1$ and $g$ is defined as

$$V(g, 1) = D \int_0^1 e^{r(t-1)}(\cos 2\pi t + 1) \, dt = D \left( \frac{gr}{r^2 + 4\pi^2} + \frac{g}{r} \right).$$  \hspace{1cm} (60)$$

Let the parameters in Eq. (58), $r$ and $g$, be fuzzy numbers. The future value of the semi-sinusoidal cash flows as in Fig. 6 is given by

$$f_{Nk}(y|\bar{F}_N) = f_k(y|\bar{D})\varphi(f_{3-k}(y|\bar{\rho}), f_k(y|\bar{g}))\varphi(f_k(y|\bar{\rho}), N),$$  \hspace{1cm} (61)

where $\varphi(r, g) = \pi(2 + g)/(r^2 + \pi^2)$, $\varphi(r, N) = (e^{rN} - 1)/(e^r - 1)$.

The present value of the semi-sinusoidal cash flows is given by Eq. (58):

$$f_{Nk}(y|\bar{P}_N) = f_k(y|\bar{D})\varphi(f_{3-k}(y|\bar{\rho}), f_k(y|\bar{g}))\psi(f_{3-k}(y|\bar{\rho}), N),$$  \hspace{1cm} (62)

where $\psi(r, N) = (e^{rN} - 1)/((e^r - 1)e^{rN})$.

The present and future values of the fuzzy cosinusoidal cash flows can be given by Eqs. (63) and (64), respectively:

$$f_{Nk}(y|\bar{P}_N) = f_k(y|\bar{D})\zeta(f_{3-k}(y|\bar{\rho}), f_k(y|\bar{g}))\psi(f_{3-k}(y|\bar{\rho}), N),$$  \hspace{1cm} (63)

where

$$\zeta(r, g) = \left( \frac{gr}{r^2 + 4\pi^2} + \frac{g}{r} \right)$$

and the fuzzy future value is

$$f_{Nk}(y|\bar{F}_N) = f_k(y|\bar{D})\zeta(f_{3-k}(y|\bar{\rho}), f_k(y|\bar{g}))\varphi(f_k(y|\bar{\rho}), N).$$  \hspace{1cm} (64)$$

See Fig. 7.

---

Fig. 5. Cosinusoidal wave cash flow function.

Fig. 6. Fuzzy semi-sinusoidal wave cash flow diagram.
10. Conclusions

In this paper, probabilistic cash flows and capital budgeting techniques in the case of fuzziness and discrete compounding have been studied. The cash flow profile of some investments projects may be geometric or trigonometric. For these kind of projects, the fuzzy present, future, and annual value formulas have been also developed under discrete and continuous compounding in this paper. Fuzzy set theory is a powerful tool in the area of management when sufficient objective data have not been obtained. Appropriate fuzzy numbers can capture the vagueness of knowledge. The other financial subjects such as replacement analysis, income tax considerations, continuous compounding in the case of fuzziness can be also applied [12,13]. Comparing projects with unequal lives has not been considered in this paper. This will also be a new area for a further study.

Appendix A

One of the most basic concepts of fuzzy set theory which can be used to generalize crisp mathematical concepts to fuzzy sets is the extension principle. Let $X$ be a Cartesian product of universes $X = X_1, \ldots, X_r$, and $\tilde{A}_1, \ldots, \tilde{A}_r$ be $r$ fuzzy sets in $X_1, \ldots, X_r$, respectively. $f$ is a mapping from $X$ to a universe $Y$, $y = f(x_1, \ldots, x_r)$. Then the extension principle allows us to define a fuzzy set $\tilde{B}$ in $Y$ by [14].

$$\tilde{B} = \{(y, \mu_{\tilde{B}}(y)) | y = f(x_1, \ldots, x_r), (x_1, \ldots, x_r) \in X\}, \quad (A.1)$$

where

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup_{(x_1, \ldots, x_r) \in f^{-1}(y)} \min \{\mu_{\tilde{A}_1}(x_1), \ldots, \mu_{\tilde{A}_r}(x_r)\}, & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise}, \end{cases} \quad (A.2)$$

where $f^{-1}$ is the inverse of $f$. 

Fig. 7. Fuzzy cosinusoidal wave cash flow diagram.
Assume $\tilde{P} = (a, b, c)$ and $\tilde{Q} = (d, e, f)$, $a, b, c, d, e, f$ are all positive numbers. With this notation and by the extension principle, some of the extended algebraic operations of triangular fuzzy numbers are expressed in the following.

**Changing sign:**

$$-(a, b, c) = (-c, -b, -a)$$

(A.3)

or

$$-(d, e, f) = (-f, -e, -d).$$

(A.4)

**Addition:**

$$\tilde{P} \oplus \tilde{Q} = (a + d, b + e, c + f)$$

(A.5)

and

$$k \oplus (a, b, c) = (k + a, k + b, k + c)$$

(A.6)

or

$$k \oplus (d, e, f) = (k + d, k + e, k + f)$$

(A.7)

if $k$ is an ordinary number (a constant).

**Subtraction:**

$$\tilde{P} - \tilde{Q} = (a - f, b - e, c - d)$$

(A.8)

and

$$(a, b, c) - k = (a - k, b - k, c - k)$$

(A.9)

or

$$(d, e, f) - k = (d - k, e - k, f - k)$$

(A.10)

if $k$ is an ordinary number.

**Multiplication:**

$$\tilde{P} \otimes \tilde{Q} = (ad, be, cf)$$

(A.11)

and

$$k \otimes (a, b, c) = (ka, kb, kc)$$

(A.12)

or

$$k \otimes (d, e, f) = (kd, ke, kf)$$

(A.13)

if $k$ is an ordinary number.

**Division:**

$$\tilde{P} \oslash \tilde{Q} = (a/f, b/e, c/d).$$

(A.14)
References