Chapter 16

Control of a Flexible Robot Arm using a Simplified Fuzzy Controller

A flexible robot arm is a distributed system per se. Its dynamics are very complicated and coupled with the non-minimum phase nature due to the non-collocated construction of the sensor and actuator. This gives rise to difficulty in the control of a flexible arm. In particular, the control of a flexible arm usually suffers from control spillover and observation spillover due to the use of a linear and approximate model. The robustness and reliability of the fuzzy control have been demonstrated in many applications, particularly, it is perfect for a nonlinear system without knowing the exact system model. However, a fuzzy control usually needs a lot of computation time. In order to alleviate this restraint, a simplified fuzzy controller is developed for real-time control of a flexible robot arm. Furthermore, the self-organizing control based on the simplified fuzzy controller is also developed. The simulation results show that the simplified fuzzy control can achieve the desired performance and the computation time is less than 10 ms so that the real-time control is possible.

16.1 Introduction

Modeling and control of flexible robot arms have been actively investigated for several years. In the past, the Euler-Bernoulli beam model was frequently used to model a one-link flexible arm and various control strategies were proposed to compensate for beam vibration [2,3,5,6,7,20,21,28]. Even the finite element model was applied to model the flexible arms [4,18]. The major difficulty in the control of a flexible arm arises from its flexible nature. Basically, a flexible arm is an infinite dimension and nonlinear system per se, while most existing modeling and control techniques are based on the finite dimension and linear model. Hence, those techniques should compensate for control spillover and observation spillover. In addition, a flexible arm with end-point feedback is a non-minimum phase system. Namely, the system has unstable zeros due to the non-collocated sensors and actuators [7,11]. Hence, the feedforward control based on the inverse dynamics is not directly applicable. In contrast with the control of the human arm, the control of the flexible robot arm seems awkward. Since the control of the human arm is based on the
sensory feedback and knowledge base, this reminds us the knowledge base control may be useful for the control of the flexible robot arm. The fuzzy control is one kind of expert controls [1]. The structure of fuzzy control is rather simple in comparison with other knowledge base control systems. Thus, it is perfect for a nonlinear system without knowing the exact system model. The robustness and reliability of the fuzzy control have been demonstrated in many applications [14,15,17]. However, fuzzy control usually takes up a lot of computation time. Although quantization and look-up table can reduce the computation time, they give rise to other problems. The quantization leads to worse precision, while the look-up table is only adequate to a special controlled process and can not be changed unless the table is set up again. In particular, when the system is large (too many system states and rules) and complex, the memory requirement becomes excessively large. Hence, the research in fuzzy hardware systems [26] and fuzzy memory devices [25] was developed. Alternatively, the simplification in the fuzzy control algorithm may lead to less computation time as well as less cost. Therefore, a simplified fuzzy controller is developed for real-time control purpose. The basic properties of the simplified fuzzy control and its relation to PID control will be addressed. Furthermore, the self-organizing control based on the simplified fuzzy controller is also developed so that the system dynamics and performance can be continuously and automatically learned and improved. Finally, the developed controllers are applied to the flexible robot arm. The results show that the simplified fuzzy control can achieve the desired performance and the computation time is less than 10 ms so that the real-time control is possible. The organization of the chapter is as follows. Section 1 describes the model of an one-link flexible arm followed by the simplified fuzzy control and the self-organizing fuzzy control. Then, the simulation results of the fuzzy control for a flexible arm are presented and followed by conclusion.

16.2 Modeling of the Flexible Arm

The one-link flexible arm is considered and shown in Figure 1. It is a thin uniform beam of stiffness EI and mass per unit length \( \rho \). The total length of the arm is \( l \) and the torque actuation point is at \( x = l \). Let \( I_B \) and \( I_H \) denote the moment of inertia of the beam and the hub, respectively. \( I_T \) is the sum of \( I_B \) and \( I_H \). The external applied torque at the joint is \( \tau \). Using the Euler-Bernoulli beam model and neglecting structural damping, the dynamic equations can be derived from Hamilton principle.

The kinetic energy of the beam and the hub is given by

\[
T = \frac{1}{2} \left[ I_H \dot{\theta}^2 + \int_0^l \rho \left( \frac{\partial y}{\partial t} \right)^2 \, dx \right] = \frac{1}{2} \sum_{i=0}^{\infty} I_T \dot{q}_i^2 ,
\]  

(16.1)

where \( \theta \) is the joint angle; \( q_i(t) \) are time-dependent modal coordinates; and \( y(x,t) \) is the displacement of the arm at the distance \( x \) from the joint. From Figure 1, \( y(x,t) \)
is given by

\[ y(x, t) = w(x, t) + x \theta , \tag{16.2} \]

where \( w \) is the deflection of the beam at the distance \( x \). The potential energy stored in the beam is given by

\[
V = \frac{1}{2} \left[ \int_0^l EI \left( \frac{\partial^2 y}{\partial x^2} \right)^2 dx - \tau \theta \right] = \frac{1}{2} \sum_{i=0}^{\infty} I_T \omega_i^2 q_i^2 + \tau \sum_{i=0}^{\infty} \phi_i^{(1)}(0) q_i , \tag{16.3}
\]

where \( \phi_i(x) \) are the normalized mode shape functions, \( \phi_i^{(1)} \) is the first derivative of \( \phi_i \), \( \tau \) is the applied torque, and \( \omega_i \) is the angular frequency. The complete solution of \( y(x, t) \) is characterized by

\[
y(x, t) = \sum_{i=0}^{\infty} \phi_i(x) q_i(t) . \tag{16.4}
\]

The partial differential equations, the boundary conditions and the detailed derivation can be found in [16]. For simplicity, only first \( n + 1 \) modes in the dynamic equations are considered and the higher modes are assumed negligible. In fact, the fuzzy controller need not use an exact model of the system. Then the dynamic equations are derived with respect to the mode shape functions and modal coordinates \( q_i \) as

\[
\ddot{q}_i + \omega_i^2 q_i = \frac{\tau}{I_T} \phi_i^{(1)}(0), \quad i = 0, 1, \ldots, n . \tag{16.5}
\]

Figure 1  Schematic diagram of a one-link flexible arm

---

where

- \( u(x, t) \): displacement measured from tangent line
- \( \rho \): mass per unit length
- \( EI \): rigidity of arm
- \( I_B \): moment of inertia about the root
- \( l \): length of arm
- \( I_H \): inertia of hub
- \( \tau(t) \): external applied torque
- \( y(x, t) \): total displacement measured from reference line
In a physical system, there exist damping forces, such as transverse velocity damping forces and strain velocity damping moments. These damping forces are considered in Rayleigh model \[19\]. In fact, the damping coefficients in Rayleigh model are characterized by the corresponding natural frequency and the coefficients of damping force and damping stress. Therefore, a damping term should be added to Equation (16.5). The resultant equation is:

\[
\ddot{q}_i + 2\zeta_i\omega_i \dot{q}_i + \omega_i^2 q_i = \frac{T}{I_T}\phi_i(1)(0), \quad i = 0, 1, \ldots, n,
\]  

(16.6)

where the damping coefficient \(\zeta_i\) can be determined by experiments. In addition, the output of the system we wish to control is the displacement at the end-point, which is given by

\[y(l, t) = \sum_{i=0}^{n} \phi_i(l)q_i(t).\]  

(16.7)

From Equations (16.6) and (16.7), the linear state space model of the flexible arm can be denoted as:

\[
\begin{align*}
\dot{Q} &= AQ + B\tau \\
\quad y &= DQ
\end{align*}
\]  

(16.8)

where

\[
\begin{align*}
Q &= [q_0 \quad \dot{q}_0 \quad q_1 \quad \dot{q}_1 \cdots q_n \quad \dot{q}_n]^T \\
D &= [l \quad 0 \quad \phi_1(l) \cdots \phi_n(l) \quad 0] \\
B &= \frac{1}{I_T}[0 \quad 1 \quad 0 \quad \phi_1^{(1)}(1) \cdots 0 \quad \phi_n^{(1)}(0)]^T
\end{align*}
\]

\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 1 & \cdots & \cdots & \cdots \\
0 & 0 & -\omega_1^2 & -2\zeta_1\omega_1 & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \vdots & \vdots & \cdots & 0 & 1 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \omega_n^2 -2\zeta_n\omega_n
\end{pmatrix}.
\]

Note that if the viscosity and deformation in the motor are considered, then \(A(2,1)\) and \(A(2,2)\) elements should not be equal to zero.

### 16.3 Simplified Fuzzy Controller

A typical fuzzy control system is shown in Figure 2.
The fuzzy controller is constituted of five main parts: scaling, fuzzification, inference engine, defuzzification, and rule base. From the aspect of controller design, three fundamental parts should be decided first. They are system states selections, rule base constitution, and shapes of membership functions. As to the constitution of rule base, some static properties, such as completeness, interaction, and consistency of control rules, must be considered [8,9,10,12,13,22,27]. Given the membership functions, what kinds of shapes (basically a fuzzy number) are adequate? There is still no systematic procedure to make the optimal decisions due to the heuristic factors among them. The most common used functions are trapezoid and triangular shapes because of the simplicity and linearity. Normal distribution function and rational polynomial function [10] are also adopted frequently because the shapes can be tuned easily and meaningfully by some parameters in the functions. Since nonlinearities exist mainly in reasoning and defuzzification, the highly coupled variables cause the difficulties in analysis. In order to have a convenient way to look into fuzzy control, some simplified procedures are proposed. At the beginning, the specifications and descriptions about the controller are stated:

(1) System States:

The controller input and output can be defined as follows. Suppose that there are \( n \) rules for a SISO control system and the connections between rules are the linguistic word ‘or’. One of the rules is expressed as below

\[
\text{rule } i : \text{if } E \text{ is } E_i \text{ and } C \text{ is } C_i \text{ then } U \text{ is } U_i, i = 1, \ldots, n,
\]

where \( E \) stands for the error between the set point and the plant output; \( C \) stands for the change of error; \( U \) stands for the control input; \( E_i, C_i, \text{ and } U_i \) are linguistic values (fuzzy sets) and belong to collections of reference fuzzy sets \( R_E = \{ E_i, i = 1 \cdots p \} \), \( R_C = \{ C_i, i = 1 \cdots q \} \), and \( R_U = \{ U_i, i = 1 \cdots r \} \), respectively.

(2) Reference Fuzzy Sets And Membership Functions:
The triangular type membership function is chosen because of its linearity. The collections of the reference fuzzy sets for the error, the change of error, and the control input are the same. The linguistic meaning of the reference fuzzy sets and the corresponding labeled numbers are listed below:

<table>
<thead>
<tr>
<th>linguistic term</th>
<th>meaning</th>
<th>antecedents</th>
<th>consequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>NB</td>
<td>Negative Big</td>
<td>0</td>
<td>-3</td>
</tr>
<tr>
<td>NM</td>
<td>Negative Medium</td>
<td>1</td>
<td>-2</td>
</tr>
<tr>
<td>NS</td>
<td>Negative Small</td>
<td>2</td>
<td>-1</td>
</tr>
<tr>
<td>ZO</td>
<td>Zero</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>PS</td>
<td>Positive Small</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>PM</td>
<td>Positive Medium</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>PB</td>
<td>Positive Big</td>
<td>6</td>
<td>3</td>
</tr>
</tbody>
</table>

All the definitions are shown in Figure 3, where $x$ axis is either “error”, “change of error”, or ”control input”.

Figure 3  Definitions of the reference fuzzy sets, membership functions, and universes of discourses

(3) Rule Base:

The type of rules used here have the following form:

if $E$ is $i$ and $C$ is $j$ then $U$ is $f(i,j)$,                        \hfill (16.11)

where $i$, $j$, and $f(i,j)$ are numeric numbers in Equation (16.10). The total rule numbers are 49($7 \times 7$). It is clear that the rule setup turns to find a proper mapping $f$. We call these kind of rules parameterized rules. Note that the mapping $f$ is also a function of another parameters; hence, we can adjust rules by tuning these parameters. For example,

$$f(i,j) = k_1 \times (i - 3) + k_2 \times (j - 3).$$  \hfill (16.12)
(4) Fuzzy Implication (t-norm, and t-conorm)

Fuzzy implication is selected as t-norm, and t-norm is chosen as an algebraic product operator; i.e.,
\[ x \triangleleft y = x \times y \, . \]  \hspace{1cm} (16.13)

Although the corresponding t-conorm can be derived by De Morgan’s law, under the consideration of defuzzification we use an approximate method instead of t-conorm to connect rules. It will be discussed in detail in the next section.

16.3.1 Derivation of Simplified Fuzzy Control Law

Let \( e \) and \( c \) be the values sensed from the plant output and scaled into the universe of discourses, respectively. As \( e \) and \( c \) are exact values, the following equation can be applied for reasoning:

\[
\mu_{U_i'}(w) = \sup\{\mu_{E_i} T \mu_{C_i}(c_0), 0\} T \mu_{U_i}(w) 
= \mu_{E_i}(e_0) T \mu_{C_i}(c_0) T \mu_{U_i}(w), 
\]  \hspace{1cm} (16.14)

where \( U' \) is the resultant fuzzy set rather than a value. A transformation called defuzzification which is opposite to the fuzzification procedure is needed to transform \( U' \) into a real value. Now, consider Figure 4. There are four rules fired:

\[
\begin{align*}
\text{if } E \text{ is } i \text{ and } C \text{ is } j \text{ then } U \text{ is } f(i,j) \, ; \\
\text{if } E \text{ is } i \text{ and } C \text{ is } j + 1 \text{ then } U \text{ is } f(i,j + 1) \, ; \\
\text{if } E \text{ is } i + 1 \text{ and } C \text{ is } j \text{ then } U \text{ is } f(i + 1,j) \, ; \\
\text{if } E \text{ is } i + 1 \text{ and } C \text{ is } j + 1 \text{ then } U \text{ is } f(i + 1,j + 1) \, .
\end{align*} 
\]  \hspace{1cm} (16.15) \hspace{1cm} (16.16) \hspace{1cm} (16.17) \hspace{1cm} (16.18)

Figure 4  Computation of membership values
From the geometric relation in Figure 4, the membership values can be easily obtained as:

\[
me(i) = \frac{D_e - d_e}{D_e} \quad (16.19)
\]

\[
me(i + 1) = \frac{d_e}{D_e} \quad (16.20)
\]

\[
cmp(j) = \frac{D_c - d_c}{D_c} \quad (16.21)
\]

\[
cmp(j + 1) = \frac{d_c}{D_c} \quad (16.22)
\]

From Equation (16.14), the contribution of the rule (16.15), for example, will have the shape as shown in Figure 5.

![Figure 5 Control inferred by the rule (i,j)](image)

The next step is to connect the fired rules and then proceed the defuzzification procedure so that a real control signal is produced. In order to obtain a simple result but not to violate “fuzzy spirit,” we use the method called weighted area procedure to obtain the real control input. The control can be computed by the following formula

\[
u = \frac{\sum_{i,j} (area) \times (center \ of \ gravity)}{\sum_{i,j} (area)} = \frac{D_u \times \sum_{i,j} me(i) \times mc(j) \times f(i, j)}{\sum_{i,j} me(i) \times mc(j)} \cdot (16.23)
\]

Note that

\[
\sum_{i,j} me(i) \times mc(j) = 1.
\]

Thus, for the firing rules (16.15) to (16.18) the control (not scaled) is

\[
u = D_u \times [me(i) \times mc(j) \times f(i, j) + me(i) \times mc(j + 1) \times f(i, j + 1) + me(i + 1) \times mc(j) \times f(i + 1, j) + me(i + 1) \times mc(j + 1) \times f(i + 1, j + 1)].
\]

(16.24)
Equation (16.24) is called the simplified fuzzy control law.

16.3.2 Analysis of Simplified Fuzzy Control Law

The simplified fuzzy control law shown in Equation (16.24) is, in fact, a nonlinear function of the error and the change of error. The expression provides a way to reduce the efforts of fuzzy computation. If the controller has $n$ inputs, then the total summation terms in the right-hand side of Equation (16.24) are $2^n$. When $n > 3$, we can see that it is not practical to apply this control law because too many terms in Equation (16.24) cost a lot of computation time. On the other hand, if $n$ is large, then the product of the membership values is small and the resultant value can not reflect the real situation. In this section, we discuss the controller with three inputs and one output. Let $e$, $c$, $s$, denote the error, the change of error, and the sum of error (scaled), respectively. There are three controller inputs. The definition is similar to that in Figure 3. The rules are still parameterized rules, i.e.,

$$\text{if } E \text{ is } i \text{ and } C \text{ is } j \text{ and } S \text{ is } k \text{ then } U \text{ is } f(i, j, k).$$

(16.25)

The control law now is

$$u = D_u \times \sum_{i,j,k} me(i) \times mc(j) \times ms(k) \times f(i, j, k),$$

(16.26)

where

$$ms(k) = \frac{D_s - d_s}{D_s},$$

$$ms(k + 1) = \frac{d_s}{D_s}.$$

The $e$, $c$, $s$ can be expressed as

$$e = (i - z) \times D_e + d_e$$

$$c = (j - z) \times D_c + d_c$$

$$s = (k - z) \times D_s + d_s$$

Then

$$d_e = e + (z - i) \times D_e$$

(16.27)

$$d_c = c + (z - j) \times D_c$$

(16.28)

$$d_s = s + (z - k) \times D_s$$

(16.29)
where

\[
    i = \sum_{m=1-z}^{z-1} u_s(e + m \times D_e)
\]

\[
    j = \sum_{m=1-z}^{z-1} u_s(c + m \times D_c)
\]

\[
    k = \sum_{m=1-z}^{z-1} u_s(k + m \times D_s)
\]

and \(u_s(\cdot)\) is a unit step function. Substituting Equation (16.27) to (16.29) into Equation (16.26), we obtain

\[
    u = D_u(c_1e + c_2c + c_3s + c_4ec + c_5cs + c_6ces + c_7ecs + c_8), \tag{16.30}
\]

where

\[
    c_1 = [(j - z + 1)(k - z + 1)(f(i + 1, j, k) - f(i, j, k))
    + (j - z + 1)(z - k)(f(i + 1, j, k + 1) - f(i, j, k + 1))
    + (z - j)(k - z + 1)(f(i + 1, j + 1, k) - f(i + 1, j, k))
    + (z - j)(z - k)(f(i + 1, j + 1, k + 1) - f(i + 1, j, k + 1))]/De, \tag{16.31}
\]

\[
    c_2 = [(i - z + 1)(k - z + 1)(f(i, j + 1, k) - f(i, j, k))
    + (i - z + 1)(z - k)(f(i, j + 1, k + 1) - f(i, j, k + 1))
    + (z - i)(k - z + 1)(f(i + 1, j + 1, k) - f(i + 1, j, k))
    + (z - i)(z - k)(f(i + 1, j + 1, k + 1) - f(i + 1, j, k + 1))]/Dc, \tag{16.32}
\]

\[
    c_3 = [(i - z + 1)(j - z + 1)(f(i, j + 1, k) - f(i, j, k))
    + (i - z + 1)(z - j)(f(i, j + 1, k + 1) - f(i, j + 1, k))
    + (z - i)(j - z + 1)(f(i + 1, j, k + 1) - f(i + 1, j, k))
    + (z - i)(z - j)(f(i + 1, j + 1, k + 1) - f(i + 1, j + 1, k))]/Ds, \tag{16.33}
\]

\[
    c_4 = [(z - k)[f(i + 1, j + 1, k + 1) - f(i + 1, j, k + 1)]
    - f(i, j + 1, k + 1) + f(i, j, k + 1)]
    + (k - z + 1)[f(i + 1, j + 1, k) - f(i + 1, j, k)]
    - f(i, j + 1, k) + f(i, j, k)]/[D_e \times D_c], \tag{16.34}
\]
\[ c_5 = \{(z - j)[f(i + 1, j + 1, k + 1) - f(i, j + 1, k)] - f(i, j + 1, k + 1) \]
\[ + f(i, j + 1, k)] + (j - z + 1)[f(i + 1, j, k + 1) - f(i + 1, j, k)) \]
\[ - f(i, j, k + 1) + f(i, j, k)] \}/[D_e \times D_e], \tag{16.35} \]

\[ c_6 = \{(z - i)[f(i + 1, j + 1, k + 1) - f(i + 1, j + 1, k)] - f(i + 1, j, k + 1) \]
\[ + f(i + 1, j, k)] + (i - z + 1)[f(i, j + 1, k + 1) - f(i, j + 1, k)) \]
\[ - f(i, j, k + 1) + f(i, j, k)] \}/[D_e \times D_e], \tag{16.36} \]

\[ c_7 = [f(i + 1, j + 1, k + 1) - f(i + 1, j + 1, k) - f(i + 1, j, k + 1) \]
\[ + f(i + 1, j, k) + f(i, j + 1, k + 1) + f(i, j, k) \]
\[ + f(i, j, k + 1) - f(i, j, k)] \}/[D_e \times D_e \times D_s], \tag{16.37} \]

\[ c_8 = (i - z + 1)(j - z + 1)(k - z + 1)f(i, j, k) \]
\[ + (i - z + 1)(j - z + 1)(z - k)f(i, j, k + 1) \]
\[ + (i - z + 1)(z - j)(k - z + 1)f(i, j + 1, k) \]
\[ + (z - i)(j - z + 1)(k - z + 1)f(i + 1, j, k) \]
\[ + (z - i)(j - z + 1)(z - k)f(i + 1, j + 1, k + 1) \]
\[ + (z - i)(z - j)(k - z + 1)f(i + 1, j + 1, k) \]
\[ + (z - i)(z - j)(z - k)f(i + 1, j + 1, k + 1) \] \tag{16.38} \]

Note that the control \( u \) in Equation (16.30) is a nonlinear function of states \( e, c, s \). Is it possible to be linear under certain conditions? Equation (16.30) is a linear controller if the nonlinear terms vanish; i.e., \( c_4, c_5, c_6, c_7 \) are equal to zero. Note that what we can manipulate is the mapping \( f \); hence, \( f \) must satisfy some restrictions so as to achieve the above intention. Consider coefficients in Equations (16.31) to (16.37). In Equations (16.31) to (16.33), if we let

\[ f(i + 1, j, k) - f(i, j, k) = \alpha \]
\[ f(i + 1, j, k + 1) - f(i, j, k + 1) = \alpha \]
\[ f(i + 1, j + 1, k) - f(i, j + 1, k) = \alpha \]
\[ f(i + 1, j + 1, k + 1) - f(i, j + 1, k + 1) = \alpha \] \tag{16.39} \]

and

\[ f(i, j + 1, k) - f(i, j, k) = \beta \]
\[ f(i, j + 1, k + 1) - f(i, j, k + 1) = \beta \]
\[ f(i + 1, j + 1, k) - f(i + 1, j, k) = \beta \]
\[ f(i + 1, j + 1, k + 1) - f(i + 1, j, k + 1) = \beta \] \tag{16.40} \]
and
\begin{align*}
    f(i, j, k + 1) - f(i, j, k) &= \gamma \\
    f(i, j + 1, k + 1) - f(i, j + 1, k) &= \gamma \\
    f(i + 1, j, k + 1) - f(i + 1, j, k) &= \gamma \\
    f(i + 1, j + 1, k + 1) - f(i + 1, j + 1, k) &= \gamma
\end{align*}
(16.41)

then Equations (16.39) to (16.41) guarantee that \( c_4, c_5, c_6, \) and \( c_7 \) vanish. The remaining coefficients become:
\begin{align*}
    c_1 &= \frac{\alpha}{D_e} \\
    c_2 &= \frac{\beta}{D_c} \\
    c_3 &= \frac{\gamma}{D_s}
\end{align*}
(16.42-16.44)

\( c_8 = \alpha(z - i) + \beta(z - j) + \gamma(z - k) + f(i, j, k) \) 
(16.45)

All the equations from Equations (16.39) to (16.41) can be understood from Figure 6.

Figure 6 Rule cube to show the relations of Equations (16.39), (16.40), and (16.41)

The final result becomes
\[ u = D_u(c_1 e + c_2 c + c_3 s + c_8) . \] 
(16.46)
It is intuitive and reasonable that there always exists a rule with the following form:

\[
\text{if } E \text{ is } ZO \text{ and } C \text{ is } ZO \text{ and } S \text{ is } ZO \text{ then } U \text{ is } ZO .
\]

This means

\[
f(z, z, z) = 0 \tag{16.47}
\]

and

\[
c_8 = 0 . \tag{16.48}
\]

Hence,

\[
f(i, j, k) = \alpha(i - z) + \beta(j - z) + \gamma(k - z) . \tag{16.49}
\]

It is interesting that Equation (16.49) can be regarded as the mapping required. In fact, from the simulation result, the rule base which is specified by the mapping (16.49) is effective and reasonable, especially for linear systems. The control law now is exact a PID controller

\[
u = D_u(c_1 e + c_2 c + c_3 s) . \tag{16.50}
\]

The above results are summarized in the following theorem.

**Theorem 1**

The simplified fuzzy control (SFC) law Equation (16.30) behaves like a PID controller if the increment of the mapping \( f \) in each direction is constant. In addition, if \( e, c, s \) fall into intervals \((-De, De), (-Dc, Dc), (-Ds, Ds)\), respectively, then the fuzzy controller is a pure PID controller.

### 16.3.3 Neglected Effect in Simplified Fuzzy Control

In the last section, we have mentioned that the weighted area method is only a convenient means in order to get the result Equation (16.24). There, \( t \) conorm is a maximum operator and center of gravity procedure is used. But, what are the side effects of a simplified fuzzy controller? Owing to the regular order of the reference fuzzy sets, there are only two possible overlap situations, as shown in Figure 7.
\[ G = (f + \frac{1}{3} + \frac{a}{3(a+b)}) \times D \]
\[ A = \frac{ab}{2(a+b)} \times D \]

Figure 7  Possible overlap conditions

Clearly, the shaded areas are the neglected parts in the simplified fuzzy control. The area and gravity of the shaded area can be obtained from the simple geometric computations. Therefore, the control law has the following expression

\[ u_g = \frac{u - \frac{1}{D_a} \sum (A \times G)}{1 - \frac{1}{D_u} \sum A} \]  \hspace{1cm} (16.51)

where \( u \) is the simplified fuzzy control law. The additional minus terms in Eq. (16.51) depends on membership values and rules fired. These terms are not easy to regulate when we try to analyze the controller. However, if the overlap among the membership functions are not significant, the simplified fuzzy control is almost the same as the original fuzzy control but with much less computation effort.

16.4 Self-Organizing Fuzzy Control

The kernel of fuzzy control is the rule base. In other words, a fuzzy system is characterized by a set of linguistic rules. The rule base is termed as the fuzzy model of the controlled process. From the point of view of traditional control, the determination of the rule base is equivalent to system identification [23]. The rules can be acquired in many ways. In the last section, parameterized rules are used. Although parameterized rules are easy to use, the tuning of parameters are trial and error and tedious. Procyk and Mamdani [24] proposed a structure of a self-organizing fuzzy controller (SOC) and simulated it by fuzzy relation (quantization approach). Basically, we will follow the idea of Mamdani with slight modification.
The block diagram of SOC proposed here is shown in Figure 8. Three additional blocks other than the fuzzy controller in Figure 8 are the reference model, the incremental model, and the parameter decision. They constitute the rules modification procedures. Under normal condition, the modification operates at every sample instant. The idea of modification procedures is stated below.

(1) The reference model senses the plant output and reference input, then compares them with the desired responses (or trajectory). The model output is the performance measure which shows how good the controller works. For simplicity, the performance measure is the change in the desired output.

(2) The incremental model is a simplified linear model of the plant. It can be derived by linearization techniques or experiments. The model relates the input change of the plant to the output change of the plant; hence, it can be used to evaluate the required control input. The incremental model does not have to be accurate because its principal function is to reflect the approximate behavior of the controlled plant.

(3) The parameter decision unit receives the information from the incremental model and proceeds with the decision making or parameter estimation. It maybe involve heuristic or mathematical approaches, or a combination of both. As to the mathematical approach, it uses the estimation technique described in adaptive control. The heuristic approach needs more a priori knowledge and depends on the designers.

For the case of two controller inputs, we have had the control law as

\[
\begin{align*}
    u &= G_u [me(i)mc(j)w(i,j) + me(i)mc(j+1)w(i,j+1) \\
    &\quad + me(i+1)mc(j)w(i+1,j) + me(i+1)mc(j+1)w(i+1,j+1)] \\
    &\quad + \Delta w \\
\end{align*}
\]

or

\[
    u = G_u (c_1 e + c_2 c + c_3 ec + c_4). 
\]

Figure 8  Structure of the self-organizing control system
Because of the parameterized rule base, the undetermined factors except scaling factors in the control law are $w(i, j)$ in Equation (16.52) or $c_i$ in Equation (16.53). Thus, the first work is to estimate coefficients $w(i, j)$ or $c_i$ via some mathematical or heuristic processes, called rules modification procedure, at every one (or two more) sampling instants. The reference model, incremental model, and parameter decision are further described below.

16.4.1 Reference Model

In model reference adaptive control, the reference model is a prescribed mathematical model and the objective is to design the control law so that the closed-loop transfer function is close to the reference model. In fact, it is a pole placement procedure. In fuzzy control, the reference model proposed here is somewhat different from MRAC (model reference adaptive control). It is a fuzzy model and not only specifies the desired response but also gives the required quantity for adjustment. The fuzzy model is all due to control objective and has nothing to do with the controlled plant.

The reference model is created as meta rules. These meta rules are used to supervise the response of plant output and give the amount of modification. The meta rule has the form as

\[
\text{if error is PS and slope is NM then change of plant output is PM}.
\]

Note that the error is defined as

\[
e(k) = \text{setpoint} - y(k) \quad (16.54)
\]
and the slope is defined as
\[ m(k) = \frac{y(k) - y(k - 1)}{T}, \]  
(16.55)
where \( T \) is the sampling period. The dimension in Equation (16.55) should be transformed into degree
\[ \text{deg}(k) = \frac{180 \times \tan^{-1}(m(k))}{\pi}. \]  
(16.56)
The meta rule base is listed in Table 1. The corresponding notations are defined in Equation (16.10). Note that zero elements in the rule table are the desired response regions, while the others need to be modified. The meta rules bear no relation to the controlled plant. They are based on the control objective. The universe of discourses and reference fuzzy sets are shown in Figure 9.

The required change of the output can be obtained again through the simplified fuzzy computation as:
\[ y_d = G_y \left[ \frac{(D_1 - d_1)(D_2 - d_2)R(i, j) + (D_1 - d_1)d_2R(i, j + 1)}{D_1D_2} \right. \\
+ \left. \frac{d_1(D_2 - d_2)R(i + 1, j) + d_1d_2R(i + 1, j + 1)}{D_1D_2} \right]. \]  
(16.57)

### 16.4.2 Incremental Model

Consider the following SISO system
\[ \dot{x} = f(x, u), \quad x \in \mathbb{R}^n \]  
(16.58)
\[ y = g(x). \]  
(16.59)
Variations in \( x \) and \( u \) cause variation in \( \dot{x} \)
\[ \delta \dot{x} = \delta f(x, u) = \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial u} \delta u. \]  
(16.60)
Note that
\[ \delta \dot{x} = \frac{d}{dt}(\delta x). \]  
(16.61)
Let \( T \) be the sampling period. Then, Equation (16.60) becomes
\[ \Delta x \approx T \delta \dot{x} = T \frac{\partial f}{\partial x} \Delta x + T \frac{\partial f}{\partial u} \Delta u \]
\[ \Delta x = (I - T \frac{\partial f}{\partial x})^{-1}T \frac{\partial f}{\partial u} \Delta u. \]  
(16.62)
From Equation (16.59), we have
\[
\delta y = \frac{\partial g}{\partial x} \delta x.
\]
(16.63)

Therefore,
\[
\Delta y = \frac{\partial g}{\partial x} \Delta x = M \Delta u
\]
\[
M = \frac{\partial g}{\partial x}(I - T \frac{\partial f}{\partial x})^{-1} T \frac{\partial f}{\partial x}
\]
(16.64)

and
\[
\Delta u = N \Delta y
\]
(16.65)

where \( N = M^{-1} \) is called the incremental value (or a matrix for MIMO case). Some remarks are made for the incremental model.

(i) For a linear system
\[
\dot{x} = Ax + Bu, \; x \in \mathbb{R}^n.
\]
(16.66)
\[
y = Cx.
\]
(16.67)

Then
\[
N = [C(I - TA)^{-1}TB]^{-1} \text{ is a constant.}
\]
(16.68)

(ii) For an unknown plant, the Jacobian matrices \( \frac{\partial f}{\partial x} \), \( \frac{\partial g}{\partial x} \), etc., can be approximately obtained from experiments. Indeed, the incremental model is a linearization of a nonlinear system. Various operating points result in various incremental value \( N \).

(iii) There is a convenient way to compute the incremental value, i.e.,
\[
N = \frac{u(k) - u(k - 1)}{y(k) - y(k - 1)}
\]
(16.69)

and take the average value from experiments or simulations.

(iv) The existence of the inverse of \( I - T \frac{\partial f}{\partial x} \) is trouble. The possible way to avoid the singular condition is that we can adjust sampling period \( T \) under the stability and the implementation of digital systems.
Table 1 Meta Rule Base

\[ e(k) \]

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(a) linguistic definition

\[ e(k) \]

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</table>

(b) corresponding labeled numbers in (a)
16.4.3 Parameter Decision

From Equations (16.57) to (16.65), we have the required change of the control input $\Delta u_d$. Hence, from Equation (16.52), we have

$$u_d(k) = u(k - 1) + \Delta u_d.$$  \hspace{1cm} (16.70)

However, only the knowledge of the value $\Delta u_d$ can not decide four weights $w(\cdot, \cdot)$ in Equation (16.52) or four coefficients $c_i$ in Equation (16.53). Since we have already known that

$$\sum_{i,j} m_e(i) m_e(j) = 1$$  \hspace{1cm} (16.71)

we obtain

$$\Delta w(i, j) = \frac{\Delta u_d}{G_u}.$$  \hspace{1cm} (16.72)

Namely,

$$w_d(i, j) = w(i, j) + \frac{\Delta u_d}{G_u}.$$  \hspace{1cm} (16.73)

Equation (16.73) is used to adjust the weights.

16.5 Simulation Results

The simulation results of the fuzzy control of the one-link flexible arm are presented in this section. The parameters of the flexible arm are listed in Table 2. Four modes are selected (including rigid mode); thus, the state space model is eight orders. The four vibration modes are listed in Table 3. The motor is modeled as a second order system, the parameters are also shown in Table 2. The simulations are performed for cases with and without the motor dynamics. In the simulation, the desired end-point position is 3 meters.

Table 2 Flexible Arm and dc Motor Parameters

Euler-Bernoulli beam:

- $l = 1 \ M$
- $E = 2 \times 10^{11} \ N/M^2$
- $\rho = 0.8 \ kg/M$
- $I = 2.5 \times 10^{-11} \ M^4$
- $A = 10^{-2} \ M^2$
- $I_H = 0.1 \ kg \cdot M^2$
- $I_B = \frac{\rho l^3}{3} \ kg \cdot M^2$
- $I_T = I_B + I_H$
DC Motor:

- Torque constant $K_i = 1.1 \, N \cdot M/Amp$
- Back emf constant $K_b = 1 \, V/rad/s$
- Rotor inertia of motor $J = 0.09 \, kg \cdot M$
- Armature resistance $R_a = 3.8 \, \Omega$
- Armature inductance $L_a = 10 \, mH$
- Viscous frictional coefficient $B = 0.02 \, N \cdot M/rad/s$

<table>
<thead>
<tr>
<th>Table 3 Four Vibration Modes of Euler Beam</th>
</tr>
</thead>
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<tr>
<td>vibration mode</td>
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<tr>
<td>natural freq. (Hz)</td>
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<tr>
<td>$\phi_i(l)$</td>
</tr>
<tr>
<td>$\phi_i^1(0)$</td>
</tr>
<tr>
<td>damping factor</td>
</tr>
</tbody>
</table>

First, the simulation results of the simplified fuzzy controller vs. conventional fuzzy controller are presented in Figure 10, Figure 11, and Figure 12. In Figure 10, the motor dynamics are not considered. The solid line denotes the simplified fuzzy controller with parameterized rule base. The dash line denotes the conventional fuzzy controller with min-max principle and center of gravity. The shapes of membership functions are adjustable (trapezoid). In contrast, the motor dynamics are considered in Figure 11. From Figure 10 and Figure 11 we can see that negative position takes place in the transient response. That is the nature of non-minimum phase system. Physically, because the beam is flexible, when the external torque is applied to the root of the beam in the beginning, the beam bends back relative to the tangent line (or rigid mode) then the tip position becomes negative. In order to reduce the negative position, the control rules in the beginning of the response should not be the original rule base. The other rule base is used for the situation that tip position is negative. The result is given in Figure 12. Clearly, the negative position of the tip has been reduced but the settling time becomes a little bit longer. Although the second rule base is applied, the phenomenon of the negative position can not be completely eliminated. The positive displacement part of the tip is acted only by the original rule base. Therefore, its response is similar to the previous figures.

The comparisons of these controllers are listed in Table 4. Obviously, the simplified fuzzy control is 40 times faster than the conventional fuzzy control. Since the computation times of the simplified fuzzy control are all within 10 ms (sampling period), the real time control can be achieved. Note that the computation time includes the computation of the mathematical model of the flexible arm; thus, these values are conservative. In addition, the use of two rule bases can reduce the negative position at the expense of longer settling time, more computation time, and larger steady state error, although they are not seriously reflected to the response.
Table 4 Comparison of Simplified and Conventional Fuzzy Controllers

<table>
<thead>
<tr>
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<th>Steady state error (m)</th>
<th>number of rules</th>
<th>computation time (ms)</th>
<th>settling time (s)</th>
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<tr>
<td>Figure 10 (Simplified)</td>
<td>0.0055</td>
<td>49</td>
<td>8.5</td>
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<td>Figure 10 (Conventional)</td>
<td>0.0168</td>
<td>49</td>
<td>350</td>
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<td>Figure 11 (Simplified)</td>
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<td>Figure 11 (Conventional)</td>
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<td>Figure 12</td>
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<td>9.3</td>
<td>1.98</td>
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</table>

Next, let us consider the case of self-organizing fuzzy control (SOC). The learning results for the flexible arm are shown in Figure 13. Figure 13 (a) is the SOC with two learning sampling periods at the first run, and the remaining runs are given in Figure 13 (b). The response at the first run is bad because we give the rules arbitrarily. Although the learning results are not perfect, the convergence is verified. In order to improve the overshoots and negative positions, a more reasonable and efficient updating law is needed.

From the simulation results of the flexible arm, we can see that the coefficients of function which tune the rules are the same, \( k_1 = 0.6 \) and \( k_2 = 0.6 \); i.e.,

\[
f(i, j) = k_1 \times (i - 3) + k_2 \times (j - 3).
\] (16.74)

In fact, for linear plants the rules are 'linear' and have similar form. Because the flexible arm is modeled as a linear system, the linear function Equation (16.74) can create a reasonable rule base for the linear plant. Table 5 lists two rule tables. Table 5(a) is obtained by inspecting step response, and Table 5(b) is created from Equation (16.74). It is clear that the two rule bases are basically the same. Table 5(b) has more rules than Table 5(a). Too many rules \( 7 \times 7 = 49 \) are the disadvantage of the parameterized rule base because some conflicts may exist among rules. Note that from Table 5 the simplified fuzzy controller behaves like a PD controller at the regions that satisfy the following relation:

\[
f(i + 1, j + 1) - f(i + 1, j) - f(i, j + 1) + f(i, j) = 0.
\] (16.75)

At the neighborhood of origin in the phase plane, the four fired rules satisfy Equation (16.75), and therefore the controller is a pure PD controller.

16.6 Conclusions

This chapter presented a simplified fuzzy controller for the one-link flexible arm. The control law is due to the simplification on reasoning and defuzzification. It offers a convenient way to compute the control input and reduces the reasoning time.

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In fact, a fuzzy controller is a highly nonlinear PD controller (if only the error and the change of error are feedback). From the simplified control law, we can see that the nonlinear term remains in the product of the error and the change of error, and the simplified fuzzy controller behaves exactly like a linear PD controller if the rules satisfy certain conditions. If the overlap between membership functions (depending on rule base) is not serious, the complete control law can be approximated by the simplified control law. This means that the fuzzy controller behaves like a variable coefficients PD controller with slight nonlinearity. The overlap of membership functions associated with the rule base constitutes the variable coefficients in the control law, and this results in robustness.

The flexible arm is a non-minimum phase system. Morris and Vidyasagar [19] showed that the Euler beam model cannot be stabilized by a finite-dimensional controller (rational function) from the viewpoint of controller design. But the fuzzy controller is a rule base system, it does not care about which mathematical model is chosen. In addition, it is a nonlinear controller; hence, the resultant closed-loop system is nonlinear. The facts shown in [19] are not adequate in these conditions. From the simulation results by the simplified fuzzy control, the tip position can be controlled well.

A self-organizing fuzzy controller using the simplified fuzzy control law was also presented in this chapter. Although the simulation results show that the justification is simple and the response converges gradually after many times learning; however, the rules’ justification seems to be coarse and unreasonable because the four fired rules are adjusted in the same weight at parameter decision procedure. There should be some heuristics to adjust the fired rules individually.

For a complex process, we may use the simplified fuzzy control as a ‘pre-controller’ so as to obtain the coarse structure of the fuzzy controller. According to the coarse controller, the work on rules’ acquirement and membership function shapes adjustment may not be so tedious.
Table 5 Comparison of Rule Tables

(a) lists the rules derived from the investigation of step response

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</table>

(b) lists the rules created from Equation (16.74)

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Figure 10  Simplified vs. conventional fuzzy controllers without motor dynamics

Figure 11  Simplified vs. conventional fuzzy controllers without motor dynamics

Figure 12  Simplified vs. conventional fuzzy controller with two rule bases
Figure 13  SOC for the 2nd system with two learning sampling periods, (a) first run, (b) remaining runs
References


